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Preface

This textbook is devoted to the study of those topics that are traditionally considered to constitute the subject of college algebra. Although a course in elementary algebra is a minimum prerequisite, the book will be most effective with those students who have also completed intermediate algebra and plane trigonometry.

The importance of a sound, thorough course in college algebra cannot be overemphasized. It is a common experience to have weaknesses in algebra hamper a student in his later study of analytic geometry and calculus. For this reason I have given primary attention to exposition in the presentation of the material and have laid particular stress on motivation and clarity of statement. It is worthy of note that every chapter begins with an introductory section whose object is to give the student not only a preliminary idea of the nature of the chapter but also its relation to later topics and other branches of mathematics.

A sincere attempt has been made to present the subject matter so as to meet most effectively the requirements of both instructor and student. The material has been arranged so that it may be readily divided into lessons for individual assignments while maintaining unity of development. Important results are presented in the form of completely stated theorems and summaries. This makes the book convenient for future reference.

The text is flexible in approach; thus courses of varying lengths may be constructed by selecting specific chapters and parts of chapters without affecting the continuity of the subject.

Greater stress has naturally been laid on the later and more advanced chapters of the text. However, the earlier chapters are more than a review of elementary and intermediate algebra; they present the subject from a more advanced standpoint, which the student is better able to comprehend in view of his greater experience.

Certain special features of the book should help to relate the material to modern developments. To begin with, Chapter 1 provides an informal, elementary introduction to the foundations, structure, and nature of algebra. The material of this chapter is substantially the contents of the opening lecture that I have given for many years to my classes in college algebra.

Chapter 2 gives a complete treatment of the very important subject of algebraic operations. Elementary proofs of the properties of the fundamental operations are presented simply and attractively at a level of rigor that is readily understood by the student. Also, at an appropriate place in this chapter, one section is devoted to an elementary introduction to the important concept of number field.

The treatment of inequalities (Chapter 6) is somewhat fuller than in most textbooks on college algebra. Past experience has shown that this is a topic where a great many students need considerable help.

A complete treatment of complex numbers is given in Chapter 8. In addition, the important concepts of group and vector are introduced and discussed in an elementary manner. The chapter concludes with an illuminating section on functions of a complex variable.

Permutations and combinations (Chapter 13) and the important subject of probability (Chapter 14) are discussed in considerable detail, with particular attention to the binomial expansion.

The subject of determinants (Chapter 15) has received special treatment. Since this topic usually presents difficulties to the student, the approach is slow and simple and places emphasis on the techniques of evaluating determinants. After the student has learned *how* to operate with determinants, he is in a much better position to understand and appreciate the proofs of the theorems.

A distinct feature of the book is the exercises, of which there are more than 2000. In addition, there are over 200 examples with complete solutions. These exercises are far more than those of the drill variety. They have been designed to accomplish a number of purposes. Primarily, of course, the exercises serve to further and complete the student's understanding of both principles and applications. In point of difficulty, the exercises range from those which are quite simple to those which represent a challenge. Some exercises have been included to introduce additional topics which the instructor may expand at his discretion.

For their unceasing help, cooperation, and encouragement, I wish to express sincere appreciation to my friends and colleagues, Professor James N. Eastham and Mr. Alan Wayne. Each has read the entire manuscript independently and painstakingly and each has contributed immeasurably to the value of the book by his comments, suggestions, and constructive criticism.

I also wish to thank those members of the editorial staff of John Wiley and Sons who have given their constant help and cooperation.

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Charles H. Lehmann

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1

Fundamental concepts

1.1. INTRODUCTION

The student who starts a course in college algebra has previously studied elementary algebra where the emphasis was mainly on algebraic operations and solutions. Little or no attention was paid to the foundations, structure, and nature of algebra. It is, therefore, the purpose of this chapter to consider some of these fundamental concepts of algebra.

The sections which follow give an elementary exposition of the distinguishing characteristics of algebra and the foundations on which the subject rests. The treatment will necessarily be brief and somewhat summary in character, for a detailed study of the structure of algebra, on a logical and rigorous basis, properly belongs to more advanced treatises. In the following discussion of the fundamental concepts, the student may draw freely on his previous experience in elementary algebra.

1.2. THE FOUNDATIONS OF ALGEBRA

In mathematics each distinct branch has a logical structure built up from certain fundamental statements which are in the nature of assumptions known as *postulates*. The student has already seen an instance of this in his study of elementary geometry. There, from a starting point consisting of certain undefined terms, definitions, and postulates, the properties of geometric figures are derived in the form of theorems, each theorem being a logical consequence of one or more preceding theorems and postulates. Similarly, the foundations of algebra rest on certain fundamental postulates, undefined terms, and definitions, as we shall now proceed to show.

The starting point of a mathematical science is associated with the meaning of certain basic words or expressions. A word is *defined* by describing it in terms of other words which are either capable of further description or else are assumed to be known. It is evident that this process will eventually lead us to a word or words for which no definitions are available. It then becomes necessary to *assume* that such words have meanings which we agree to accept without formal definition. It is at this point that we lay the basis for a logical science such as algebra.

Since there are no restrictions at the beginning, we are free to choose those terms which we agree to accept without definition. It is natural and customary to restrict such selection to the simplest and most fundamental concepts which, moreover, will not subsequently lead to any contradictions. The student will recall that his *first* experience with computation was in counting the number of objects in a group. For this purpose he used certain symbols designated as 1, 2, 3, 4, $\cdots$, and called them numbers. For our purposes, we will refer to such numbers as the *positive integers*. Accordingly, we now state

POSTULATE 1. We *assume* the existence of the *positive integers*, employed in counting the number of objects in a group, and designated by the symbols 1, 2, 3, 4, $\cdots$.

The next step in the student's experience with computation was the determination of the *total number* of objects in two or more distinct groups. This required an operation called *addition*. In particular, to determine the *total number* of objects in two or more groups of *equal size*, an operation called *multiplication* was employed. These two fundamental and basic operations are the motivation for

POSTULATE 2. There are two *operations* on the positive integers, called *addition* and *multiplication* and designated, respectively, by the symbols $+$ and $\times$.

With these two postulates as a starting point, it is possible to create the entire number system of algebra, as outlined in the next section.

1.3. THE NUMBER SYSTEM OF ALGEBRA

If the operations of addition and multiplication are performed upon positive integers, the results are again positive integers. Evidently, then, the two basic postulates of algebra (Sec. 1.2) restrict all computation to positive integers and to the two operations of addition and multiplication. To remove this restriction and to meet our requirements for other numbers, for example, negative numbers and fractions, it becomes necessary to introduce further concepts.

In elementary algebra the student learned to use letters of the alphabet to represent numbers. Accordingly, let a and b represent two given positive integers which we are to add together, and let c represent their *sum*. We then write the relation

$$a + b = c \tag{1}$$

and state that it represents the *solution* of the problem: *Given* two positive integers a and b, *find* their sum c. Now consider a *converse* of this problem, namely, *given* the sum c of two positive integers a and b, and *given* the positive integer a, *find* the other positive integer b. The solution of this problem requires an operation *inverse* to the operation of addition and called *subtraction*. This new operation is represented by the symbol $-$, and we write the solution in the form

$$b = c - a, \tag{2}$$

which states that b is the result of *subtracting* a from c. From his previous experience with numbers the student will realize that relations (1) and (2) are equivalent, either one being obtainable from the other.

We now note the important fact that in a number system restricted to positive integers, it is impossible to subtract a larger number from a smaller number. To make subtraction possible in this case, we introduce new numbers called the *negative integers* and designated by the symbols $-1, -2, -3, \cdots$. In particular, if we subtract an integer from itself, we have the important number *zero* designated by the symbol 0. Thus, if a represents any integer, we have the relation

$$a - a = 0, \tag{3}$$

which we may regard as a *definition* of zero. Note that zero is neither a positive nor a negative integer.

We next consider the postulated operation of multiplication. Let a and b represent two given integers which we are to multiply together, and let c represent their *product*. We then write the relation

$$a \times b = c, \tag{4}$$

where a and b are called the *factors* of c, and state that it represents the *solution* of the problem: *Given* two integers a and b, *find* their product c. Now consider a *converse* of this problem, namely, *given* the product c of two integers a and b, and *given* the factor a, *find* the other factor b. The solution of this problem requires an operation *inverse* to the operation of multiplication and called *division*. We write the solution in the form

$$b = \frac{c}{a}, \tag{5}$$

which states that b is the result of *dividing* c by a. In relation (5), c is called the *dividend*, a the *divisor*, and b the *quotient*.

It is important to note that in a number system restricted to integers, it is not always possible to perform the operation of division. Thus, if we divide the integer 6 by the integer 3, the result is 2, another integer. But if we attempt to divide the integer 5 by the integer 3, the operation fails because no integer exists such that when multiplied by the integer 3, the product is the integer 5. To make division possible in this case, we introduce new numbers called *fractions* and represented by the right member of relation (5) where the integer c is called the *numerator* and the integer a is called the *denominator*.

With fractions included in our number system, the operation of division, as expressed by relation (5), is possible in all cases except one, namely, when the divisor a is zero. We shall see later that *division by zero is an excluded operation*. We note further that relations (4) and (5) are equivalent, either one being obtainable from the other, provided, however, that *the divisor a is different from zero* in relation (5).

At this point our number system consists of the positive and negative integers, zero, and the positive and negative fractions. These numbers constitute the *rational number system* for which we have the

Definition. A number is said to be *rational* if it can be expressed in the form p/q where p is any positive or negative integer or zero, and q is any positive or negative integer.

The integers are rational numbers. For example, $5 = \frac{5}{1} = \frac{10}{2}$, etc. Also, zero is a rational number since $0 = 0/a$ where a is any integer and therefore different from zero.

We next consider a special case of multiplication where the factors to be multiplied together are all equal. Thus, if we multiply the number a by itself, we have the product aa, which we usually write in the form a^2. In general, the product of n factors, each equal to a, is written in the form a^n where the positive integer n is called the *index* or *exponent*. We then say that we are *raising the number a to the nth power* and call this operation *involution*.

Let us now write this operation in the form

$$a^n = b, \tag{6}$$

which represents the solution of the problem: *Given* the number a and the positive integer n, *find* the number b, the nth power of a. We now consider the *converse* of this problem, namely, *given* the number b and the positive integer n, *find* the number a whose nth power is equal to b. The solution of this problem requires an operation *inverse* to the operation of involution and called *evolution*. We write the solution in the form

$$a = \sqrt[n]{b}, \tag{7}$$

which states that a is an nth *root of* b. For this reason the operation of evolution is also called the *extraction of a root*. In relation (7), the symbol $\sqrt{\ }$ is called the *radical sign* and the integer n is called the *index of the root*.

We have now arrived at an important stage in the development of the number system of algebra. The operations of addition, subtraction, multiplication, division, and involution, when applied to rational numbers, produce unique results which are also rational numbers, that is, there is no further extension of our number system. This, however, is not necessarily true for the operation of evolution. For example, the square root of 4 is not a unique number but may be either $+2$ or -2 since $(-2)^2 = 4$ as well as $(+2)^2$. In this case, the results, while not unique, are still rational numbers. However, let us now consider the positive square root of 2, which may be written simply as $\sqrt{2}$. It is not difficult to show that this number cannot be expressed in the form p/q to meet the requirements of our definition of a rational number. Such a number is then said to be *irrational*. The rational number system, together with all the positive and negative irrational numbers, constitute the *real number system* of algebra.

We shall now investigate the last extension of our number system. We have seen that the operation of evolution would not be possible in some cases if we were restricted to the rational number system. To meet this situation we added the irrational numbers to our number system. We note also that our examples above involved the square root of positive numbers only. But to make evolution possible in all cases, we must consider the extraction of roots of negative numbers as well. For example, let us attempt to find the square root of -4, that is, we wish to find a number a such that $a^2 = -4$. But a fundamental property of the real number system is that the square (or even power) of any real number (positive or negative) is a positive real number. Evidently the number a does not have this property and therefore cannot belong to the real number system. To meet this situation we must introduce a new number.

Let c be any positive number so that $-c$ is a negative number and $\pm\sqrt{-c}$ is not a real number. We may write

$$\pm\sqrt{-c} = \pm\sqrt{c}\sqrt{-1}. \tag{8}$$

In this relation, $\pm\sqrt{c}$ is a real number, so that if we are to give any meaning to $\pm\sqrt{-c}$, we must give a meaning to $\sqrt{-1}$. For this purpose we have the

Definition. The quantity $\sqrt{-1}$ is called the *imaginary unit*, is represented by the symbol i, and has the property that $i^2 = -1$.

In view of this definition, we may write relation (8) in the form

$$\pm\sqrt{-c} = \pm\sqrt{c}i.$$

Since $\pm\sqrt{c}$ is a real number, we may represent it by the real number b so that bi represents a new class of numbers for which we have the

Definition. A number of the form bi, where b is any real number and i is the imaginary unit, is called a *pure imaginary number*.

Later we will have particular use for a number expressed as the sum of a real number and a pure imaginary number. Accordingly we have the

Definition. A number of the form $a + bi$, where a and b are real numbers and i is the imaginary unit, is called a *complex number*.

In view of our previous statements, we may now say that in order to make all six operations possible in all cases, we have extended our number system so as to include the complex numbers. But we may also make a very significant observation about the general complex number $a + bi$. If $a = 0$ but $b \neq 0$, $a + bi$ assumes the form bi, so that the pure imaginary number is a special case of the complex number. If $b = 0$, $a + bi$ assumes the form a, and hence represents a real number. From this viewpoint, a real number is merely a special case of a complex number and, accordingly, the set of all real numbers is said to be a *subclass* of the system of complex numbers. Although we will often have occasion to make a precise distinction between real and complex numbers, we shall, in view of this last statement, consider *the system of complex numbers to be the number system of algebra*.

1.4. THE OPERATIONS OF ALGEBRA

The six operations discussed in the previous section—addition, subtraction, multiplication, division, involution, and evolution—are the operations of algebra. These operations are of extreme importance, not only in algebra but also in all other branches of mathematics where they may be employed. They are used subject to certain restrictions or conditions called *laws*. It is absolutely essential to apply the operations properly in accordance with these laws in order to obtain correct results. The improper use of algebraic operations has probably caused students more trouble, not only in algebra but also in other branches of mathematics, than anything else. Hence we see the obvious importance of the correct application of the algebraic operations and why the entire next chapter is devoted to this topic.

We noted in the previous section that, in order to make the operations of algebra possible in all cases, it was necessary to extend our original postulated number system of the positive integers by introducing, in turn, the negative integers, zero, fractions, irrational numbers, and, finally, complex numbers. The student may now naturally pose the question: If any of the six operations of algebra are applied to complex numbers, will it be necessary to introduce any new type of number different from the complex number? The answer is no. We shall see later that the application of the operations of algebra to complex numbers always results in complex numbers. We then say that the complex number system of algebra is *closed* under the six operations of algebra, that is, the complex number system is adequate for the application of all algebraic operations.

1.5. THE STRUCTURE OF ALGEBRA

It is impossible to give a concise and yet satisfactory answer to the question: What is algebra? Any such attempt will fall far short of giving the student an adequate conception of the subject. But we are now in a position to state that algebra has a structure which is characterized by

(1) A specified set of symbols representing complex numbers.
(2) A specified set of operations on the symbols (1)—the six operations of algebra.
(3) The laws of the operations (2).

The first two items have been considered in Secs. 1.3 and 1.4, respectively; item (3) will be discussed in the next chapter.

Evidently, then, algebra has a very simple structure. We shall see subsequently that the various topics and problems considered in algebra are those which involve subjecting the symbols (1) to the operations (2) in accordance with the laws (3).

1.6. THE NATURE OF ALGEBRA

It is customary and natural to present the subject of algebra initially to the student as an extension or generalization of arithmetic. The student then becomes acquainted with negative numbers for the first time. He also learns to use the letters of the alphabet to represent numbers and soon realizes the advantage of solving certain problems by letting the letter x or some other letter represent the unknown quantity. We now see that these concepts are examples of the structure of algebra.

We may summarize the material of this chapter by giving a characterization of the *nature of algebra* in the following

Fundamental definition. A mathematical process is said to be *algebraic* if it involves one or more of the operations of addition, subtraction, multiplication, division, involution, and evolution applied one or more times in any order to any complex numbers or to any symbol or symbols representing complex numbers.

As an example of this definition, consider the expression $2x^2 - 3xy + 4y^2$. This expression is said to be *algebraic* because it has been formed by applying algebraic operations to numbers and letters which represent numbers.

As another example, consider the quadratic equation

$$ax^2 + bx + c = 0, \qquad a \neq 0.$$

The student who has studied this equation will recall that its solution is given in the form

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This solution is algebraic since it involves algebraic operations on numbers. It is interesting to note that all six algebraic operations are used in this solution.

We conclude by briefly discussing another topic which throws further light on the nature of algebra.

When the student studied geometry in high school, he was told that the subject is known as *Euclidean* geometry. He may also have been told that there are other geometries, known as *non-Euclidean* geometries, with properties markedly different from those of Euclidean geometry. Similarly, as we shall now see, there are other algebras with properties different from those of the algebra we are studying.

There are numbers whose structure is different from, or a generalization of, our complex number $a + bi$. Such numbers are called *hypercomplex numbers*. One type of hypercomplex number is known as the *quaternion*. Since a quaternion differs from an ordinary complex number, we might expect them to differ in various ways. We may note one such difference. Ordinary complex numbers obey the *commutative law of multiplication*, which states that the product of two numbers is independent of the order of multiplication. Thus, if x and y are two complex numbers, the product xy is identical to the product yx. However, if A and B are two quaternions, it is not true, in general, that AB and BA are equal. The properties and applications of quaternions constitute the field of study known as the *algebra of quaternions*.

Another algebra of considerable interest is the *algebra of matrices*. The basic form in this algebra is called a *matrix;* it is of great importance in modern mathematics and physics. Like quaternions, matrices do not, as a rule, obey the commutative law of multiplication.

There are many other algebras besides the two noted above, but their discussion is beyond the scope of this text and will be found in advanced treatises. In contradistinction to these algebras, the subject we are about to study in this book is often referred to as the *algebra of complex numbers*.

2

Algebraic operations

2.1. INTRODUCTION

This chapter will deal with the operations of algebra (Sec. 1.4) and the manner in which they are performed. Once again we emphasize the great importance of carrying such operations through correctly. The ability to manipulate algebraic expressions accurately and expeditiously is a prime prerequisite for satisfactory progress in the applications of algebra. Such skills are acquired mainly by practice. The student is therefore strongly urged to solve as many as possible of the problems contained in the groups of exercises in this chapter.

2.2. ALGEBRAIC EXPRESSION, TERM, POLYNOMIAL

In accordance with the fundamental definition of an algebraic process (Sec. 1.6), the result of such a process is called an *algebraic expression*. Thus, $3x^2y + z$ is an algebraic expression because it is obtained by applying algebraic operations to the number 3 and the letters x, y, and z, which represent numbers. Other examples of algebraic expressions are $6x^2 - 7x + 8$, $2a + \frac{\sqrt{a}}{b}$, and $\frac{x^2 + 2}{x^3 - 5x^2 + 7}$.

The simplest representation of a number is a single numeral or letter, for example, the numeral 5 or the letter b. The most compact representation of a number, employing more than a single numeral or letter, is obtained by combining these numbers and letters by any of the operations of algebra except addition and subtraction. Illustrations of such representation are the simple algebraic expressions $5xy$, $2a^2b$, $3x/2y$, $4\sqrt{ac}$. Each such compact representation of a number is called an *algebraic term*.

Any factor of an algebraic term is called the *coefficient* of the remaining factors. Thus, in the algebraic term $5xy$, 5 is the coefficient of xy and $5x$ is the coefficient of y. However, it is usually convenient to consider only one number or letter as a coefficient. Thus, we designate 5 as the (numerical) coefficient of xy in the term $5xy$ and b as the (literal) coefficient of xy in the term bxy.

Algebraic terms which differ only in their coefficients are said to be *similar*. Thus, $5xy$ and $-7xy$ are similar terms.

If the letters (or literal portion) of an algebraic term involve only the operation of multiplication (or involution), the term is said to be *rational integral*. Thus, the terms $5xy$, $-\frac{3}{2}x^2$, and $\sqrt{5}ab^2c^3$ are all rational integral. We note that the exponents in a rational integral term are positive integers. By the *degree* of such a term we mean the number representing the sum of all these exponents. Thus, $5xy$ is of degree 2, $-\frac{3}{2}x^2$ is of degree 2, and $\sqrt{5}ab^2c^3$ is of degree 6.

A single algebraic term is called a *monomial*. If two or more algebraic terms are connected by plus and (or) minus signs, the resulting expression is called an *algebraic sum*. The algebraic sum of two terms is called a *binomial* and that of three terms a *trinomial*. In general, the algebraic sum of two or more terms is called a *multinomial*. Thus, the multinomial $4x^2 - 2x\sqrt{y} + y^2/2$ consists of the terms $4x^2$, $-2x\sqrt{y}$, and $y^2/2$. It should be especially noted that *the terms of a multinomial are separated by plus or minus sign*.

The particular type of multinomial in which each term is rational integral is called a *polynomial*. Examples of polynomials are $2x^2 + 3xy + y^2$, $\sqrt{2}z^3 - \frac{4}{3}z^2 + 4z - 8$, and $3x^4 + 4x^3 - 2x^2 - 8x + 5$. By the *degree* of a polynomial we mean the highest degree of any term in it. Thus, the three polynomials just exhibited are of degrees 2, 3, and 4, respectively. If each term of a polynomial is of the same degree, it is said to be *homogeneous*. Thus the first polynomial is homogeneous, but the second and third are not.

The algebraic operations considered in this chapter will be performed on the types of algebraic expressions described above. Furthermore, unless otherwise noted we will consider these operations to be applied to *real numbers only*. Later we shall make a special study of the complex number (Chapter 8).

2.3. ADDITION

The operation of addition is characterized by the following five postulates, called *laws*.

(1) *Existence law. Addition is always possible.* That is, it is always possible to perform this operation for any two or more numbers; the result is again a number.

(2) *Uniqueness law. Addition is unique.* That is, for any two given numbers a and b, there is one and only one number c such that $a + b = c$. The unique number c is called the *sum* of a and b.

(3) *Commutative law. Addition is commutative.* That is, if a and b are any two numbers, then $a + b = b + a$. In other words, the sum of two (or more) numbers is independent of the order of summation.

Example. $2 + 5 = 5 + 2$.

(4) *Associative law. Addition is associative.* That is, if a, b, c are any three numbers, then $(a + b) + c = a + (b + c)$. In other words, the sum of three (or more) numbers is independent of the manner in which they are grouped.

Example. $(2 + 5) + 8 = 2 + (5 + 8)$.

(5) *Equality law.* If a, b, and c are any numbers such that $a = b$, then $a + c = b + c$.

The student will recognize this law as the familiar axiom that if equal numbers be added to equal numbers, the sums are equal.

These laws may be readily extended to any number of quantities.

In describing the associative law of addition, we used a *symbol of grouping* called the *parentheses* and designated by the symbol ( ). The purpose of this symbol is to indicate that all the terms enclosed are to be considered as a single number. Other symbols of grouping are the *brackets*, [ ]; the *braces*, { }; and the *bar* or *vinculum*, $\overline{\quad}$, placed above the grouped quantities as in $2 + \overline{5 + 8}$.

The addition of algebraic expressions whose terms are all positive is performed exactly as in arithmetic. If some of the terms are negative, however, the procedure requires special treatment. Since negative numbers are introduced in order to make subtraction possible in all cases (Sec. 1.3), we will defer consideration of problems in algebraic addition until after the discussion of the operation of subtraction.

2.4. SUBTRACTION

In Sec. 1.3 we described subtraction as the operation inverse to the operation of addition. Subtraction is *defined* in the following

ASSUMPTION. For any two numbers a and c, there exists one and only one number b such that

(1) $$a + b = c.$$

This number b is then given by the relation

(2) $$b = c - a,$$

which reads "b equals c minus a," and we say that b is the *difference* or *remainder* obtained by *subtracting* the *subtrahend* a from the *minuend* c.

Example. $5 + 2 = 7$, whence $2 = 7 - 5$.

We also say that b is the number which must be *added* to the number a in order to produce the number c. Thus, from (1) and (2), we have the relation

(3) $$a + (c - a) = c.$$

The operation of subtraction is subject to the

Equality law. If a, b, and c are any numbers such that $a = b$, then $a - c = b - c$.

The student will recognize this law as the familiar axiom that if equal numbers be subtracted from equal numbers, the differences are equal.

It is important to note in the above assumption that the result of subtraction is *unique*. We now investigate the question of having the operation of subtraction always possible. This involves the meaning of the statement that one number is *greater* than another. For this purpose we have the following

Definition. The number x is said to be *greater* than the number y, provided that $x - y$ is a positive number. We then write $x > y$, which is read "x is greater than y."

Example. $7 > 5$, since $7 - 5 = 2$, a positive number.

The relation $x > y$ also implies that y is *less* than x, written $y < x$. These two relations are of course equivalent.

With reference to relation (2) above, there are three cases to consider.

(I) $a < c$. Here $b = c - a$ is a positive number. This case is the familiar situation in arithmetic where we subtract a smaller from a larger number.

(II) $a = c$. Here $b = c - a = c - c = 0$ by definition of zero (Sec. 1.3). Hence, from (1) we have

$$a + 0 = a$$

whence, from the commutative law of addition (Sec. 2.3), we have

(4) $$a + 0 = 0 + a = a,$$

an important property of zero.

(III) $a > c$. In this case we have the situation where we attempt to subtract a larger from a smaller number. This is the first significant deviation from the operations of arithmetic.

From $a > c$ it follows that $a - c = p$, a positive number, so that the expression $c - a$ in relation (2) is meaningless in a system restricted to positive numbers. In order to make subtraction possible in this case, we *define* $c - a$ in relation (2) to be a *negative number* and write

$$c - a = -p,$$

so that

$$c - a = -(a - c). \tag{5}$$

As an illustration of relation (5), we have

$$5 - 7 = -(7 - 5) = -2.$$

In the particular case where $c = 0$, the defined negative number $c - a$ takes the form $0 - a$, which is abbreviated into $-a$ and called the *negative of a*. That is,

$$0 - a = -a. \tag{6}$$

The positive number p is sometimes written $+p$, read "plus p," in order to emphasize the positive sign. The negative number $-p$, read "minus p," is always preceded by the negative sign. Positive and negative numbers are often called *signed numbers*. If p is any positive number, it is convenient to refer to $-p$ as its *corresponding negative number*. Thus, -5 is the corresponding negative number of 5.

The *absolute value* of any number a, designated by $|a|$, is its numerical or ordinary arithmetical value without regard to sign. Thus $|5| = 5$ and $|-2| = 2$. Evidently any positive number and its corresponding negative number have the same absolute value.

In referring to signed numbers above, we used the positive and negative signs as *signs of quality to denote "positive number" or "negative number."* However, these signs have also been previously used as *signs of operation*. This double use or significance of the positive and negative signs is justified by the theorems which follow.

Theorem 1. *The sum of any positive number and its corresponding negative number is zero.*

PROOF. Let a be any positive number so that $-a$ is its corresponding negative number. Then, by relation (6) above,

$$a + (-a) = a + (0 - a).$$

Now in relation (3), the definition of subtraction, if we set $c = 0$, we have $a + (0 - a) = 0$, so that the right member of the preceding equation becomes zero.

Hence, $a + (-a) = 0$.

This completes the proof.

A simple illustration of this theorem is $5 + (-5) = 0$.

Theorem 2. *The operation of adding a negative number is equivalent to the operation of subtracting a positive number of the same absolute value.*

PROOF. Let a be any number, and let b represent a positive number so that $-b$ is its corresponding negative number. We are to prove that

$$a + (-b) = a - b. \tag{7}$$

By the uniqueness law of addition (Sec. 2.3), let

$$a + (-b) = c. \tag{8}$$

Adding b to both sides (equality law, Sec. 2.3),

$$[a + (-b)] + b = c + b.$$

whence $a + [(-b) + b] = c + b$ by the associative law.

By Theorem 1, $\quad (-b) + b = 0.$

Hence, $\quad a + 0 = c + b,$

and by (4), $\quad a = c + b,$

whence, by relations (1) and (2), we have

$$c = a - b. \tag{9}$$

Then from (8) and (9) we have the required result (7).

By means of Theorem 2 and the definition of subtraction, we may also establish

Theorem 3. *The operation of subtracting a negative number is equivalent to the operation of adding a positive number of the same absolute value.*

That is, if a is any number, and if b is a positive number so that $-b$ is its corresponding negative number, then

$$a - (-b) = a + b.$$

The proof of this theorem is left as an exercise to the student.

We are now in a position to give a complete statement of algebraic addition in

Theorem 4. *Let a, b, and $p = a + b$ be three positive numbers so that $-a$, $-b$, and $-p$ are their respective corresponding negative numbers. Then the following relations hold for algebraic addition:*

I. $a + b = p$.

II. $-a + (-b) = -a - b = -(a + b) = -p$.

III. If $a > b$, then $a + (-b) = a - b$.
If $a < b$, then $a + (-b) = a - b = -(b - a)$.

Relation I is arithmetic and a part of the hypothesis. Relations II and III follow from Theorem 2 and relation (5).

These relations may be stated as follows:

I and II. To add two numbers with like signs, add their absolute values and prefix their common sign to the sum.

III. To add two numbers with unlike signs, subtract the smaller absolute value from the larger absolute value and prefix the sign of the number having the larger absolute value to the difference.

The following simple examples illustrate these relations.

$$2 + 5 = 7.$$
$$(-2) + (-5) = -2 - 5 = -(2 + 5) = -7.$$
$$(-2) + (5) = -2 + 5 = 5 - 2 = 3.$$
$$(2) + (-5) = 2 - 5 = -(5 - 2) = -3.$$

The relations in Theorem 4 may be extended to three or more numbers.

As a direct consequence of Theorems 2, 3, and 4, we have the procedure for subtraction as stated in

Theorem 5. *The operation of subtracting one number from another consists in changing the sign of the subtrahend and then proceeding as in algebraic addition* (*Theorem* 4).

At this point we call attention to a simple but important property connecting positive and negative numbers and zero. Let a be a positive number so that $-a$ is a negative number. Then by relation (4) we have

$$a + 0 = a,$$

whence, by the definition of subtraction, relation (2), we have

$$a - 0 = a. \tag{10}$$

By Theorem 3, $\quad 0 - (-a) = 0 + a.$

whence, from relation (4), we have

$$0 - (-a) = a. \tag{11}$$

Now, by our previous definition of "greater than," it follows from (10) that

(12) $$a > 0,$$

and, from (11), that $0 > -a$, or

(13) $$-a < 0.$$

From relations (12) and (13) we have

Theorem 6. *A positive number is greater than zero; a negative number is less than zero.*

From this theorem it follows that *zero is neither a positive nor a negative number.* Accordingly, by the term *non-negative numbers*, we mean all positive numbers *and zero.* If a is such a number we write $a \geq 0$, which is read "a is greater than or equal to zero."

We now illustrate the operations of algebraic addition and subtraction by means of several examples.

Example 1. Find the sum of the following algebraic expressions: $x^3 + 2x^2y - 4xy^2$, $2x^3 - 4x^2y + 3y^3$, $2xy^2 - 4y^3$.

SOLUTION. We first write the expressions so that similar terms appear in the same column. Then we apply the rules of addition as given by Theorem 4. The result is the following arrangement:

$$\begin{array}{rrrrr} & x^3 + 2x^2y & - 4xy^2 & \\ & 2x^3 - 4x^2y & & + 3y^3 \\ & & 2xy^2 & - 4y^3 \\ \hline \text{Sum} = & 3x^3 - 2x^2y & - 2xy^2 & - \ y^3. \end{array}$$

Example 2. Find the remainder obtained by subtracting $a^3 - 3a^2 + 4a - 7$ from $2a^3 + a^2 - 3a - 5$.

SOLUTION. We write the subtrahend below the minuend so that similar terms appear in the same column. Then considering the sign of each term in the subtrahend to be changed, we add similar terms in accordance with Theorem 5. The work is exhibited below.

$$\begin{array}{ll} \text{Minuend} & 2a^3 + \ a^2 - 3a - 5 \\ \text{Subtrahend} & \underline{\ a^3 - 3a^2 + 4a - 7} \\ \text{Remainder} = & \ a^3 + 4a^2 - 7a + 2. \end{array}$$

The student may find it easier to actually change the sign of each term in the subtrahend and then add.

Addition and subtraction of algebraic expressions are often involved with the symbols of grouping (Sec. 2.3). The simplification of such expressions requires the removal of these symbols. From our previous results we have the following procedure for an algebraic expression enclosed in parentheses:

If preceded by the plus sign, the parentheses may be removed without any changes; if preceded by the minus sign, the parentheses may be removed, provided that the sign of each of the enclosed terms is changed.

If an expression involves more than one symbol of grouping, such symbols may be removed in any order. Often, however, it is simpler to remove one symbol at a time, removing the innermost symbol at each step.

Example 3. Simplify the expression:

$$5a - (2a - \{4a + 2b + [a - 3b]\}).$$

SOLUTION. Removing the brackets first, we have

$$\begin{aligned} &5a - (2a - \{4a + 2b + a - 3b\}) \\ &= 5a - (2a - 4a - 2b - a + 3b) \\ &= 5a - 2a + 4a + 2b + a - 3b = 8a - b. \end{aligned}$$

With practice the student may perform two or more steps at a time and shorten the simplification considerably.

EXERCISES. GROUP 1

In each of Exs. 1–5, find the sum of the given algebraic expressions.

1. $2a^3 - 2a^2b + 2b^3$, $3a^2b - 4ab^2 - 4b^3$, $2ab^2 - a^3$.
2. $4m^2 - 3mn + 2n^2$, $6mn - 2n^2 + 5$, $3n^2 - 3 - 2m^2$.
3. $x^2 - 4xy + 3y^2$, $2x^2 + 2xy - 2y^2$, $2xy - y^2 - x^2$.
4. $3x^3 - 8x^2 + 9x$, $-x^3 + 3x^2 - 8$, $2x^3 - 2x^2 - 7x + 5$.
5. $c^2 + 2cd - 2d$, $3c - 3cd - 2d^2$, $c^2 + 4d - 2c + 2d^2$.

In each of Exs. 6–10, find the remainder obtained by subtracting the second expression from the first.

6. $3a - 2b + 4c - d$, $2a + b - 3c - d$.
7. $x^3 - 4x^2 + 2x - 5$, $-x^3 + 2x^2 - 3x - 3$.
8. $a^3 - 3a^2b + 3ab^2 - b^3$, $a^3 - 4a^2b + 2ab^2 + b^3$.
9. $2a + 4by - 2cy^2 + dy^3$, $2dy^3 - 2by - a + 3cy^2$.
10. $m^4 + 6m^3 - 7m^2 + 8m - 9$, $2m^3 + 3m^2 - 4m - 3$.

In Exs. 11–15, $A = x^3 + 2x^2 - 3x + 1$, $B = 2x^3 - x^2 + 4x - 7$, and $C = x^3 + x^2 - 6x - 2$.

11. Find $A + B - C$.
12. Find $A - B + C$.
13. Find $A - B - C$.
14. Find $B - A + C$.
15. Find $B - A - C$.

16. Show that the sum of all the expressions in Exs. 11–15 is equal to the expression in Ex. 11.

In each of Exs. 17–21, simplify the given expression.

17. $5 - \{2 + 3 - (4 - \overline{3 - 2}) + [5 - 8]\}$.
18. $4 + [5 - (6 - 9 + \{7 - 2\}) - (12 - 5)]$.
19. $x + 2y - (4y - x + [3x - 2y] - \{2x - 2y\})$.
20. $4a - [6b + \{2a - [3b + a - \overline{b + 4a}]\}]$.
21. $m + 2n - \{3m - \overline{2m + n} - (2n - [m - 4n])\}$.

22. (*a*) Find the number which must be added to -8 to give a sum of 15. (*b*) Find the number which must be added to 7 to give a sum of -3.

23. (*a*) Find the number which must be subtracted from 4 to give a remainder of 6. (*b*) Find the number which must be subtracted from -11 to give a remainder of 4.

24. (*a*) Find the number which must be diminished by 8 to give a remainder of -2. (*b*) Find the number which must be diminished by -7 to give a remainder of 4.

25. Find the expression which must be added to $3a - 2b + 4c$ to give a sum of $2a + 3b - 2c$.

26. Find the expression which must be subtracted from $4x + 2y - 7$ to give a remainder of $3x - y + 5$.

27. Find the expression which must be diminished by $2m - 2n + 3p$ to give a remainder of $4m + n - 2p$.

Each of Exs. 28–31 refers to a problem in subtraction.

28. The minuend is $2a^2 + 2ab - b^2$; the remainder is $a^2 + 3ab - 2b^2$. Find the subtrahend.

29. The subtrahend is $x^2 + 3x - 7$; the remainder is $3x^2 - 3x + 4$. Find the minuend.

30. The remainder is $x^2 + 2xy - 3y^2$; the minuend is $3x^2 - 2xy + y^2$. Find the subtrahend.

31. The remainder is $a^3 + 3a^2 - 2a + 5$; the subtrahend is $2a^3 - 2a^2 + a - 5$. Find the minuend.

32. By means of the definition of "greater than," verify the following relations: $9 > 2$; $-2 > -9$; $2 > -9$.

33. If a is a positive number, verify the following relations: $-3a > -5a$; $a > -2a$; $-4a < -a$.

34. Show how the uniqueness law of addition may be extended to three or more numbers.

35. Show how the commutative law of addition may be extended to three or more numbers.

36. Show how the associative law of addition may be extended to four or more numbers.

37. Prove that the sum of any negative number and its absolute value is equal to zero.

38. Establish Theorem 3 of Sec. 2.4.

39. Give a detailed statement of the proof of Theorem 4 of Sec. 2.4.

40. Give a detailed statement of the proof of Theorem 5 of Sec. 2.4.

2.5. MULTIPLICATION

As noted in Sec. 1.2, multiplication, like addition, is a postulated operation of algebra. It is characterized by five postulates or laws analogous to those of addition (Sec. 2.3). In stating these laws, we observe that the symbol of multiplication, $\times$ or $\cdot$, is often omitted in expressing the operation for letters. Thus, $a \times b$, $a \cdot b$, and ab each have the same significance.

(1) *Existence law. Multiplication is always possible.* That is, it is always possible to perform this operation for any two or more numbers; the result is again a number.

(2) *Uniqueness law. Multiplication is unique.* That is, for any two given numbers a and b, there is one and only one number c such that $ab = c$. The unique number c is called the *product* of a and b which are termed its *factors*. The factors a and b are also called the *multiplicand* and *multiplier*, respectively.

(3) *Commutative law. Multiplication is commutative.* That is, if a and b are any two numbers, then $ab = ba$. In other words, the product of two (or more) numbers is independent of the order of multiplication.

Example. $2 \times 5 = 5 \times 2$.

(4) *Associative law. Multiplication is associative.* That is, if a, b, and c are any three numbers, then $(ab)c = a(bc)$. In other words, the product of three (or more) numbers is independent of the order in which they are grouped.

Example. $(2 \cdot 5)8 = 2(5 \cdot 8)$.

(5) *Equality law.* If a, b, and c are any numbers such that $a = b$, then $ac = bc$.

The student will recognize this law as the familiar axiom that if equal numbers be multiplied by equal numbers, the products are equal.

Multiplication and addition are connected by the important *Distributive law. Multiplication is distributive with respect to addition.* That is, if a, b, and c are any three numbers, then $a(b + c) = ab + ac$.

Example. $3(2 + 7) = 3 \times 2 + 3 \times 7$.

These laws may be extended to any number of quantities.

We shall now derive some of the fundamental properties of multiplication. We begin by establishing an extension to the distributive law.

Theorem 7. *The operation of multiplication is distributive with respect to subtraction. That is, for any three numbers a, b, c,*

$$a(b - c) = ab - ac.$$

PROOF. Let

(1) $$b - c = x,$$

whence, by the definition of subtraction (Sec. 2.4),

$$b = c + x.$$

Then, by the equality law (5),

$$ab = a(c + x),$$

whence, by the distributive law above,

$$ab = ac + ax,$$

and, by the definition of subtraction,

$$ax = ab - ac.$$

Replacing x in this last relation by its value given in (1),

$$a(b - c) = ab - ac.$$

This completes the proof.

We next establish

Theorem 8. *The product of any number and zero is equal to zero.*

PROOF. Let a be any number. Then by the definition of zero (Sec. 1.3),

$$a \cdot 0 = a(b - b)$$

By Theorem 7, $= ab - ab$

By definition of zero, $= 0$.

This completes the proof.

In the next two theorems we establish the law of signs for multiplication. For this purpose we will require the following:

ASSUMPTION. The product of two positive numbers is a positive number.

Theorem 9. *The product of a positive number and a negative number is a negative number.*

PROOF. Let a and b be any positive numbers so that $-b$ is a negative number. Let

(2) $$a(-b) = x.$$

By the equality law for addition (Sec. 2.3),

$$a(-b) + ab = x + ab,$$

whence, by the distributive law,

$$a[(-b) + b] = x + ab.$$

But, by the definition of zero, $(-b) + b = 0$.

Hence, $$a \cdot 0 = x + ab.$$

By Theorem 8, $$0 = x + ab,$$

whence, by the definition of subtraction,

$$x = 0 - ab,$$

and, by relation (6) of Sec. 2.4, $x = -ab$, so that from (2), $a(-b) = -ab$.

Now since a and b are both positive, it follows from our assumption above that their product ab is positive; thus $-ab$ is a negative number.

This completes the proof.

Theorem 10. *The product of two negative numbers is a positive number.*

PROOF. Let a and b be any two positive numbers so that $-a$ and $-b$ are negative numbers. Let

(3) $$(-a)(-b) = x.$$

Then by the equality law for addition,

$$(-a)(-b) + a(-b) = x + a(-b).$$

Then, by the distributive law and Theorem 9,

$$(-b)[(-a) + a] = x - ab,$$

whence, by definition of zero, $(-b) \cdot 0 = x - ab$.

By Theorem 8, $$0 = x - ab,$$

whence, by definition of subtraction, $0 + ab = x$,
and, by relation (4) of Sec. 2.4, $ab = x$,
so that from (3), $(-a)(-b) = ab$.

Moreover, since a and b are both positive, ab is positive by our postulate above, and the theorem is established.

From Theorems 9 and 10 we have the

Rule of Signs for Multiplication

1. The product of two numbers of like sign is positive; the product of two numbers of unlike sign is negative.
2. In general, the product of any number of factors is positive if there are either no negative factors or else an *even* number of negative factors; the product is negative if there are an *odd* number of negative factors.

Examples.

$$2 \times 5 = 10.$$
$$(2)(-5) = -(2)(5) = -10.$$
$$(-2)(-5) = +(2)(5) = 10.$$
$$(2)(-3)(-5) = +(2)(3)(5) = 30.$$
$$(2)(3)(-5) = -(2)(3)(5) = -30.$$

We are now in a position to establish a very important theorem to which we shall refer later. This theorem, the converse of Theorem 8, is stated as

Theorem 11. *If the product of two numbers is equal to zero, at least one of these numbers is equal to zero.*

PROOF. Let a and b be two numbers such that

$$ab = 0.$$

If $a = 0$, the theorem follows immediately. Suppose $a \neq 0$ (read "a is not equal to zero"); then, to prove our theorem we must show that $b = 0$. Assume, contrary to the theorem, that $b \neq 0$. Since both a and b are now assumed different from zero, each must be either positive or negative by Theorem 6 (Sec. 2.4). If they agree in sign, ab is positive; if opposite in sign, ab is negative by the rule of signs. But this contradicts our hypothesis that $ab = 0$. Hence our assumption that $b \neq 0$ is false, and the theorem is established.

Corollary. *If the product of two or more factors is equal to zero, at least one of these factors is equal to zero.*

We now consider the actual multiplication of algebraic expressions. In performing this operation we often obtain considerable economy in expressing the terms of a product by making use of certain results called *index laws* or *laws of exponents*. We have already noted, in connection with the operation of involution (Sec. 1.3), that by the notation a^n, where a is any number and n is a positive integer called the *exponent* or *index*, we mean the product of n factors each equal to a. We also refer to a^n as the nth *power of a*. In particular, the exponent 1 is generally omitted while a^2 and a^3 are also called the *square of a* and the *cube of a*, respectively. At this time we require the following three index laws, where a and b are any numbers and m and n are positive integers.

I. $$a^m a^n = a^{m+n}.$$

Example. $$2^3 \cdot 2^2 = 2^5.$$

II. $$(a^m)^n = a^{mn}.$$

Example. $$(2^3)^2 = 2^6.$$

III. $$(ab)^m = a^m b^m.$$

Example. $$(3 \cdot 2)^3 = 3^3 \cdot 2^3.$$

These index laws are very easily established. Thus, for Index Law I, we have, by the associative law of multiplication,

$$\begin{aligned} a^m a^n &= (a \cdot a \cdot a \cdots \text{ to } m \text{ factors})(a \cdot a \cdot a \cdots \text{ to } n \text{ factors}) \\ &= a \cdot a \cdot a \cdots \text{ to } m + n \text{ factors} \\ &= a^{m+n}. \end{aligned}$$

The proofs of Index Laws II and III are similar and are left as exercises to the student.

By means of these index laws and the rule of signs, we may obtain the product of two or more monomials as illustrated in

Example 1. Find the indicated products:

(a) $(2a^2b)(-3ab^2)$; (b) $(-4xy^2z)(-2x^2yz)(xyz^2)$;

(c) $(-3m^2n^3)^2$; (d) $(-2p^2q)^3$.

SOLUTION.

(a) $(2a^2b)(-3ab^2) = -6a^{2+1}\,b^{1+2} = -6a^3b^3$.

(b) $(-4xy^2z)(-2x^2yz)(xyz^2) = 8x^{1+2+1}\,y^{2+1+1}\,z^{1+1+2} = 8x^4y^4z^4$.

(c) $(-3m^2n^3)^2 = (-3)^2(m^2)^2(n^3)^2 = 9m^4n^6$.

(d) $(-2p^2q)^3 = (-2)^3(p^2)^3q^3 = -8p^6q^3$.

We next consider the product of a monomial and a polynomial. The procedure here follows immediately from the distributive law as illustrated in

Example 2. Evaluate $a^2b\,(2ax - 3by - 2ab^2)$.

SOLUTION. By the distributive law,

$$a^2b(2ax - 3by - 2ab^2) = (a^2b)(2ax) - (a^2b)(3by) - (a^2b)(2ab^2)$$
$$= 2a^3bx - 3a^2b^2y - 2a^3b^3.$$

Finally, we consider the product of two polynomials. Once again the procedure follows from the distributive law. To show this we will, for simplicity, consider the product of two binomials. Thus, by the distributive law,

$$(a + b)(x + y) = (a + b)x + (a + b)y$$

Again, by the distributive law, $= ax + bx + ay + by.$
Thus, we see, as in the case of Example 2, that the product of two expressions consists of the algebraic sum of the products of monomials obtained by multiplying each term in the multiplicand by each term in the multiplier. In actual practice it is convenient to write the multiplier under the multiplicand, each in descending powers of some letter, and then arrange the products in rows so that similar terms appear under each other to facilitate addition. This procedure is illustrated in the following example.

Example 3. Multiply $x^2 + xy - 2y^2$ by $3y^2 - 2xy + x^2$.

SOLUTION. We first write the multiplicand and multiplier in descending powers of x. The operation of multiplication then appears as follows:

$$\begin{array}{llllll}
 & x^2 + & xy & - 2y^2 & & \text{multiplicand} \\
 & x^2 - & 2xy & + 3y^2 & & \text{multiplier} \\
\hline
(1) & x^4 + & x^3y & - 2x^2y^2 & & \\
(2) & & - 2x^3y & - 2x^2y^2 & + 4xy^3 & \\
(3) & & & 3x^2y^2 & + 3xy^3 & - 6y^4 \\
\hline
\text{Product} & x^4 - & x^3y & - x^2y^2 & + 7xy^3 & - 6y^4.
\end{array}$$

Lines (1), (2), and (3) are obtained by multiplying each term in the multiplicand by x^2, $-2xy$, and $3y^2$, respectively. The product is then the algebraic sum of these three lines.

NOTES 1. The accuracy of algebraic operations may often be partially checked by substituting numerical values for the letters involved. Thus, in the preceding example, if we let $x = 2$ and $y = 3$, we have the following values:

Multiplicand $= 4 + (2)(3) - 2(9) = -8.$
Multiplier $= 4 - 2(2)(3) + 3(9) = 19.$
Product $= 16 - (8)(3) - (4)(9) + 7(2)(27) - 6(81) = -152,$

which agrees with $(-8)(19)$.

2. If both the multiplicand and multiplier are homogeneous polynomials, the product is also a homogeneous polynomial. The preceding example illustrates this.

2.6. SPECIAL PRODUCTS

We list here certain special products which are useful in various problems of multiplication and factoring. The student should memorize the following nine types, which may be established by direct multiplication.

1. $(a + b)^2 = a^2 + 2ab + b^2$.
2. $(a - b)^2 = a^2 - 2ab + b^2$.
3. $(a + b)(a - b) = a^2 - b^2$.
4. $(x + a)(x + b) = x^2 + (a + b)x + ab$.
5. $(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd$.
6. $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.
7. $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$.
8. $(a + b)(a^2 - ab + b^2) = a^3 + b^3$.
9. $(a - b)(a^2 + ab + b^2) = a^3 - b^3$.

By the use of double signs it is possible to combine certain pairs of these types in one statement. Thus, types 1 and 2 may be expressed in the single statement:

$$(a \pm b)^2 = a^2 \pm 2ab + b^2,$$

from which type 1 is obtained by using the upper signs and type 2 by using the lower signs. A similar remark applies to types 6 and 7 and types 8 and 9.

Some types may be readily expressed in words. Thus, type 1 may be stated as follows: The square of the sum of any two numbers is equal to the sum of the squares of these numbers plus twice their product.

At this point attention is called to a very important skill which the student should acquire as early as possible, namely, the ability to recognize *mathematical forms* and the generalizations and extensions obtainable from such forms. Thus, in the statement of type 1, since the operation applies to the square of the sum of *any* two numbers, such numbers may be represented in a great variety of forms, but the operation is performed in the same basic way. This is illustrated in

Example 1. Evaluate $[x^2 + 2x + y - 3]^2$.

SOLUTION. $[x^2 + 2x + y - 3]^2 = [(x^2 + 2x) + (y - 3)]^2$

By type 1, $= (x^2 + 2x)^2 + 2(x^2 + 2x)(y - 3) + (y - 3)^2$

$$= (x^4 + 4x^3 + 4x^2) + (2x^2y - 6x^2 + 4xy - 12x) + (y^2 - 6y + 9)$$
$$= x^4 + 4x^3 + 2x^2y - 2x^2 + 4xy + y^2 - 12x - 6y + 9.$$

Similarly, the student should recognize that type 3 concerns the product of the sum and difference of the *same two quantities*. An illustration of this is given in

Example 2. Find the product of $x + y - 2$ and $x - y + 2$.

SOLUTION. We can, of course, obtain the product by direct multiplication, as in illustrative Example 3 of Sec. 2.5. However, we may write

$$(x + y - 2)(x - y + 2) = [x + (y - 2)][x - (y - 2)]$$

By type 3, $= x^2 - (y - 2)^2$

By type 2, $= x^2 - (y^2 - 4y + 4)$

$$= x^2 - y^2 + 4y - 4.$$

Example 3. Evaluate $(3x^2 - 2y)^3$.

SOLUTION. By type 7, we have

$$(3x^2 - 2y)^3 = (3x^2)^3 - 3(3x^2)^2(2y) + 3(3x^2)(2y)^2 - (2y)^3$$
$$= 27x^6 - 54x^4y + 36x^2y^2 - 8y^3.$$

Finally, we consider the square of any polynomial.
By direct multiplication, we have

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

This result is a particular example of

Theorem 12. *The square of any polynomial is equal to the sum of the squares of each of its terms plus twice the product of each term by each of its following terms.*

This theorem may be established by a method of proof called *mathematical induction*, which will be studied later. The student should observe that types 1 and 2 are special cases of this theorem. He should also use this theorem to obtain the result of illustrative Example 1.

EXERCISES. GROUP 2

In each of Exs. 1–15, find the indicated product.

1. $(8a^2b)(-2ab^2)$.
2. $(-ab^2c)(3a^2bc)(2abc^2)$.
3. $xy^2(x^2 - 2y + 4)$.
4. $(2x^2 - 5y)(4x + 2y^2)$.
5. $(a^2 + 2ab - 2b^2)(3a - 7b)$.
6. $(x^2 - 3xy + y^2)(2x - 3y + 2)$.

7. $(a^2 - 2ab + 4b^2)(a + 2b)$. Check this problem by letting $a = 2$ and $b = 3$.

8. $(x^2 + y^2 + z^2 - xy - xz - yz)(x + y + z)$.

9. $(m^3 - m^2 + m - 1)(-m^3 + m^2 - m + 1)$.

10. $(2 + 3x^2 + x^3)(x^2 - 1 + 4x)$.

11. $(x + a)(y + a)(z + a)$.

12. $(x^2 - x - 1)^2(x^2 + x + 1)$.

13. $(a^4 + a^3b + a^2b^2 + ab^3 + b^4)(a - b)$.

14. $(a^2 - ab + b^2 + a + b + 1)(a + b - 1)$.

15. $(a^2 - a + 1)(a^4 - a^2 + 1)(a^2 + a + 1)$. Check this problem by letting $a = 2$.

16. Show how the uniqueness law of multiplication may be extended to the product of three or more numbers.

17. Show how the commutative law of multiplication may be extended to the product of three or more numbers.

18. Show how the associative law of multiplication may be extended to the product of four or more numbers.

19. Show how the distributive law may be extended to four or more numbers.

20. Establish the corollary to Theorem 11 (Sec. 2.5).

21. Verify the numerical examples given to illustrate Index Laws I, II, and III (Sec. 2.5).

22. Establish Index Laws II and III (Sec. 2.5).

23. By means of examples illustrate the difference between Index Laws I and II (Sec. 2.5).

24. Show that Index Law I may be extended to three or more factors, that is, show that $a^m a^n a^p \cdots a^r = a^{m+n+p+\cdots+r}$.

25. Show that Index Law III may be extended to three or more factors, that is, show that $(abc \cdots z)^m = a^m b^m c^m \cdots z^m$.

26. Show that Index Law III may be generalized in the form $(a^p b^q c^r \cdots)^m = a^{pm} b^{qm} c^{rm} \cdots$.

27. Show that the product of two homogeneous polynomials is also a homogeneous polynomial and that the degree of the product is equal to the sum of the degrees of the multiplicand and multiplier.

Exercises 28–34 refer to the nine types of special products listed in Sec. 2.6.

28. By actual multiplication, establish types 1, 2, and 3.

29. By actual multiplication, establish types 4 and 5.

30. By actual multiplication, establish types 6 and 7.

31. By actual multiplication, establish types 8 and 9.

32. Express each of types 2 and 3 in words.

33. Express each of types 6 and 7 in words.

34. By means of the use of double signs, express in one statement: (a) types 6 and 7; (b) types 8 and 9.

35. Verify the result of Example 1 of Sec. 2.6 by using Theorem 12.

In Exs. 36–50 evaluate the given expressions by means of the type forms and Theorem 12 of Sec. 2.6.

36. $(2x^2 - 3y^2)^2$.
37. $(a^2 - ab)^2$.
38. $(a^2 + 3)(a^2 - 3)$.
39. $(ax + xy)(ax - xy)$.
40. $(x^2 + x + 1)(x^2 + x - 1)$.
41. $(a - b + c)(a + b + c)$.
42. $(2x + 5)(3x - 2)$.
43. $(4x - 2)(3x + 2)$.
44. $(2c^2 + d^2)^3$.
45. $(3m - 2n^2)^3$.
46. $(a + b)^4$.
47. $(x^4 + 1)(x^2 + 1)(x^2 - 1)$.
48. $(x^2 + x + 1)(x^2 - x + 1)(x^4 - x^2 + 1)$.
49. $(a - b + c - d)^2$.
50. $(2w - x + 2y - z)^2$.

2.7. DIVISION

In Sec. 1.3 we described division as the operation inverse to the operation of multiplication. Division is *defined* in the following

ASSUMPTION. For any two numbers a and c, $a \neq 0$, there exists one and only one number b such that

$$ab = c. \tag{1}$$

This number b is then given by the relation

$$b = \frac{c}{a}, \qquad a \neq 0, \tag{2}$$

which is read "b equals c divided by a," and we say that b is the *quotient* obtained by dividing the *dividend* c by the *divisor* a.

Example. $5 \cdot 2 = 10$, whence $2 = \frac{10}{5}$.

We also say that b is the number by which the number a must be multiplied in order to produce the number c. Thus, from (1) and (2), we have the relation

$$a \cdot \frac{c}{a} = c, \qquad a \neq 0. \tag{3}$$

NOTE. In relation (2), the operation of division is indicated by a horizontal line. It may also be indicated by an oblique line or by the symbol $\div$. Thus, $\frac{c}{a}$, c/a, and $c \div a$ each have the same significance.

The operation of division is subject to the

Equality Law. If a, b, and c are any numbers such that $a = b$ and $c \neq 0$, then $a/c = b/c$.

The student will recognize this law as the familiar axiom that if equal numbers be divided by equal (nonzero) numbers, the quotients are equal.

It is important to note in the assumption above that the result of division is *unique*. It is also important to observe that division is possible in every case except when the divisor is zero. This follows from

Theorem 13. *Division by zero is an excluded operation.*

PROOF. In defining division by means of relation (1) above, namely,

$$ab = c, \tag{1}$$

we specified that the number b is unique provided that $a \neq 0$. Suppose, contrary to this definition, that $a = 0$. Since no restrictions are placed on the number c, we consider two cases.

Case (1). $c = 0$. In this case, relation (1) takes the form

$$ab = 0. \tag{4}$$

But if $a = 0$, b may be *any* number in relation (4) by Theorem 8 (Sec. 2.5). But this is contrary to our requirement of the *uniqueness of b*.

Case (2). $c \neq 0$. In this case, if $a = 0$ in relation (1), c must be equal to zero by Theorem 8 (Sec. 2.5), a contradiction.

Hence, in either case, the assumption that $a = 0$ leads to a contradiction, and the theorem follows.

The student should not infer from the preceding theorem that it is impossible to divide into zero. For this situation we have

Theorem 14. If zero is divided by any nonzero number, the quotient is zero.

PROOF. For $c = 0$ in relation (1) above, we have

$$ab = 0. \tag{4}$$

Since $a \neq 0$, it follows from Theorem 11 (Sec. 2.5) that $b = 0$. That is, in relation (2) above, $b = c/a = 0/a = 0$. This completes the proof.

From relation (1) above, for the particular case where $c = a \neq 0$, we have

$$ab = a, \tag{5}$$

whence

$$b = \frac{a}{a}.$$

In this case, the quotient b is called *unity*, and is represented by 1, the symbol for the positive integer one, and we write

$$b = \frac{a}{a} = 1,$$

whence, from (5), we have the relations

$$a \cdot 1 = a, \quad 1 \cdot a = a, \quad \text{and} \quad a = \frac{a}{1}.$$

For the particular case where $c = 1$, relation (1) gives

$$(6) \qquad ab = 1.$$

In this case the quotient b is called the *reciprocal of a*, and we write

$$b = \frac{1}{a}, \qquad a \neq 0,$$

whence from (6), $$a \cdot \frac{1}{a} = 1.$$

From these results we have the following

Properties of Unity

1. The result of multiplying or dividing any number by unity is the original number.

2. The product of any nonzero number and its reciprocal is equal to unity.

We now derive the rule of signs for the operation of division. For this purpose we refer to relations (1) and (2) above, namely,

$$(1) \qquad ab = c,$$

$$(2) \qquad b = \frac{c}{a}, \qquad a \neq 0.$$

By the rule of signs for multiplication (Sec. 2.5), if a and c are both positive, or are both negative in (1), b must be positive. Also, if a is positive and c is negative, or if a is negative and c is positive in (1), b must be negative. Then, from relation (2), we have at once the

Rule of Signs for Division

The quotient of two numbers is positive or negative according to whether the dividend and divisor have like or unlike signs.

Hence, if a, b, and c are all positive, we may write

$$b = \frac{c}{a} = \frac{-c}{-a}; \qquad -b = \frac{-c}{a} = \frac{c}{-a} = -\frac{c}{a}.$$

Theorem 15. *The product of two quotients a/b and c/d is another quotient given by the relation*

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

PROOF. By the associative and commutative laws of multiplication (Sec. 2.5), we have

$$\frac{a}{b} \cdot \frac{c}{d} \cdot bd = \left(\frac{a}{b} \cdot b\right) \cdot \left(\frac{c}{d} \cdot d\right)$$

By relation (3), $= ac,$
whence, by the equality law of division,

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

Corollary 1. $$\frac{ac}{bd} = \frac{a}{d} \cdot \frac{c}{b}.$$

Corollary 2. $$\frac{ac}{b} = \frac{a}{b} \cdot c = a \cdot \frac{c}{b}.$$

Corollary 3. $$\frac{a}{b} = \frac{a}{1} \cdot \frac{1}{b} = a \cdot \frac{1}{b}.$$

That is, *to divide by a number is equivalent to multiplying by its reciprocal.*

Also, as a consequence of Theorem 15, we have, for m a positive integer,

$$\left(\frac{a}{b}\right)^m = \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} \cdots \text{to } m \text{ factors} = \frac{a \cdot a \cdot a \cdots \text{to } m \text{ factors}}{b \cdot b \cdot b \cdots \text{to } m \text{ factors}} = \frac{a^m}{b^m}.$$

That is, to the three index laws of Sec 2.5, we now add

Index Law IV. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$, m a positive integer.

Also, at this time, we will require

Index Law V. For $a \neq 0$ and m and n positive integers such that $m > n$,

$$\frac{a^m}{a^n} = a^{m-n}.$$

For, by Index Law I (Sec. 2.5), we have

$$(a^{m-n})(a^n) = a^{m-n+n} = a^m,$$

whence, by definition of division, $a^{m-n} = \dfrac{a^m}{a^n}$.

As a direct consequence of Theorem 15 and Index Law V, we have

Theorem 16. *If a and b are both different from zero, and m, n, r, and s are positive integers such that m > n and r > s, then*

$$\frac{a^m b^r}{a^n b^s} = a^{m-n} b^{r-s}.$$

NOTE. Theorem 16 may be extended to three or more quotients.

In this section we confine the operation of division to rational integral expressions so that all exponents involved are positive. Negative and fractional exponents, and also zero as an exponent, will be considered in later sections.

We are now prepared to divide one rational integral monomial by another of lower degree, as illustrated in

Example 1. Perform the indicated divisions:

(a) $(6a^3b^2) \div (-2a^2b)$; (b) $(5x^3y^3z) \div 2x^2yz$;
(c) $(-4m^4n^3) \div (-2m^3n^2)$.

SOLUTION. From Theorems 15 and 16 we have

(a) $$\frac{6a^3b^2}{-2a^2b} = \frac{6}{-2} \cdot \frac{a^3}{a^2} \cdot \frac{b^2}{b} = -3ab.$$

(b) $$\frac{5x^3y^3z}{2x^2yz} = \frac{5}{2} \cdot \frac{x^3}{x^2} \cdot \frac{y^3}{y} \cdot \frac{z}{z} = \frac{5}{2} xy^2 \cdot 1 = \frac{5}{2} xy^2.$$

(c) $$\frac{-4m^4n^3}{-2m^3n^2} = \frac{-4}{-2} \cdot \frac{m^4}{m^3} \cdot \frac{n^3}{n^2} = 2mn.$$

We next consider the division of a polynomial by a monomial. For this operation we have

Theorem 17. *The division of a polynomial by a monomial is effected by dividing each term of the polynomial by the monomial, and adding the resulting quotients. That is,*

$$\frac{a + b + c}{m} = \frac{a}{m} + \frac{b}{m} + \frac{c}{m}.$$

PROOF. By Corollary 3 of Theorem 15,

$$\frac{a+b+c}{m} = \frac{1}{m}(a+b+c)$$

Distributive law of multiplication (Sec. 2.5),

$$= \frac{1}{m}\cdot a + \frac{1}{m}\cdot b + \frac{1}{m}\cdot c$$

Corollary 3, Theorem 15,

$$= \frac{a}{m} + \frac{b}{m} + \frac{c}{m}.$$

Example 2. Divide $2a^3bx - 3a^2b^2y - 2a^3b^3$ by a^2b.

SOLUTION. By Theorem 17, we have

$$\frac{2a^3bx - 3a^2b^2y - 2a^3b^3}{a^2b} = \frac{2a^3bx}{a^2b} - \frac{3a^2b^2y}{a^2b} - \frac{2a^3b^3}{a^2b}$$

$$= 2ax - 3by - 2ab^2.$$

This operation may be easily performed in one step.

The student should compare this problem with illustrative Example 2 of Sec. 2.5.

Finally, we consider the problem of dividing one polynomial by another. In this operation we are required to obtain an expression (the quotient) which, when multiplied into the divisor, will produce the dividend. Hence the dividend is composed of all the partial products obtained from the multiplication of the divisor by each term of the quotient. (Compare the discussion in Sec. 2.5 preceding illustrative Example 3 on the product of two polynomials.) This is the basis of the

Procedure in the Division of One Polynomial by Another

1. Arrange both dividend and divisor according to the descending powers of some common letter.
2. Divide the first term of the dividend by the first term of the divisor, and write the result as the first term of the quotient. Multiply the entire divisor by this term, and subtract the product from the dividend.
3. Consider the remainder in Step 2 as a new dividend and repeat the process of Step 2 to obtain the second term of the quotient.
4. Continue this process until the remainder is either zero or of lower degree than the divisor.

This procedure is illustrated in

Example 3. Divide $x^4 - x^3y - x^2y^2 + 7xy^3 - 6y^4$ by $x^2 + xy - 2y^2$.

SOLUTION. The operation appears as follows:

$$\begin{array}{r}
x^2 - 2xy + 3y^2 = \text{quotient} \\
x^2 + xy - 2y^2 \overline{) x^4 - x^3y - x^2y^2 + 7xy^3 - 6y^4} \\
\underline{x^4 + x^3y - 2x^2y^2} \qquad\qquad\qquad \\
-2x^3y + x^2y^2 + 7xy^3 \qquad \\
\underline{-2x^3y - 2x^2y^2 + 4xy^3} \qquad \\
3x^2y^2 + 3xy^3 - 6y^4 \\
\underline{3x^2y^2 + 3xy^3 - 6y^4}
\end{array}$$

The student will find it highly instructive to compare this operation with the corresponding operation of multiplication exhibited in the solution of Example 3 of Sec. 2.5.

When the remainder is zero, as in the above example, the division is termed *exact* and the dividend is said to be *exactly divisible* by the divisor which is then called an *exact divisor* or *factor* of the dividend.

Let A represent the dividend, B the divisor, Q the quotient, and R the remainder for the operation of division. If $R = 0$, the division is exact, and we write

$$\frac{A}{B} = Q,$$

whence

$$A = BQ.$$

This relation shows that exact division may be checked by showing that the dividend is equal to the product of the divisor and quotient.

If $R \neq 0$, the division may be made exact if the original dividend is diminished by R. We then write

$$\frac{A - R}{B} = Q,$$

whence $$A - R = BQ \text{ and}$$

(7) $$A = BQ + R.$$

Relation (7) shows that *any* operation in division may be checked by showing that the dividend is obtained by multiplying the divisor by the quotient and then adding the remainder to this product.

If we divide relation (7) through by B, we obtain

(8) $$\frac{A}{B} = Q + \frac{R}{B}.$$

Example 4. Divide $a^3 - 3a^2 + 4a - 7$ by $a^2 + a - 1$.

SOLUTION. The operation follows:

$$\begin{array}{r l}
 & a - 4 = \text{quotient} \\
a^2 + a - 1 & \overline{)a^3 - 3a^2 + 4a - 7} \\
 & \underline{a^3 + a^2 - a} \\
 & -4a^2 + 5a \\
 & \underline{-4a^2 - 4a + 4} \\
 & 9a - 11 = \text{remainder.}
\end{array}$$

In accordance with relation (8), we may write the result as:

$$\frac{a^3 - 3a^2 + 4a - 7}{a^2 + a - 1} = a - 4 + \frac{9a - 11}{a^2 + a - 1}.$$

The student should check this example by means of relation (7).

EXERCISES. GROUP 3

In each of Exs. 1–22, perform the indicated division and check the result.

1. $(8x^4y^3z^2) \div (-4x^2y^2z)$.
2. $(-15a^2m^3n^4) \div (-5am^2n^2)$.
3. $(4abx^3 - 8b^2x^2y) \div (2bx^2)$.
4. $(2a^2mx^2y) + 6a^2nyz^2) \div (2a^2y)$.
5. $(2x^2 + xy - 6y^2) \div (x + 2y)$.
6. $(x^3 - y^3) \div (x - y)$.
7. $(3a^2 - 10ab + 3b^2) \div (3a - b)$.
8. $(a^3 + b^3) \div (a + b)$.
9. $(m^4 - n^4) \div (m + n)$.
10. $(m^4 - n^4) \div (m - n)$.
11. $(x^5 + y^5) \div (x + y)$.
12. $(x^5 - y^5) \div (x - y)$.
13. $(3x^3 - 5x^2y - 8xy^2 - 2y^3) \div (3x + y)$.
14. $(a^5 - 4a^4 + 3a^3 + 3a^2 - 3a + 2) \div (a^2 - a - 2)$.
15. $(2a^4 - a^3b - 6a^2b^2 + 7ab^3 - 2b^4) \div (a^2 + ab - 2b^2)$.
16. $(2x^5 + 3x^4 - 5x^3 + 2x^2 + 7x - 6) \div (2x^2 + x - 2)$.
17. $(2x^2 + 3xy - 2y^2 - 2x + 6y - 4) \div (x + 2y - 2)$.
18. $(x^3 - 3x^2 + x - 5) \div (x - 2)$.
19. $(4a^4 + 2a^3 - 4a^2 + 3a - 7) \div (2a - 1)$.
20. $(x^3 + 2x^2 - 3x + 4) \div (x^2 - x + 2)$.
21. $(a^4 - a^3b - ab^3 + b^4) \div (a^2 + ab + b^2)$.
22. $(x^4 + 2x^3 + 3x^2 - 4x + 2) \div (x^3 + x^2 - x + 1)$.
23. Solve Ex. 14 by arranging the dividend and divisor according to the ascending powers of a.
24. Solve Ex. 16 by arranging the dividend and divisor according to the ascending powers of x.

25. Check Ex. 15 by letting $a = 2$ and $b = 1$.

26. Check Ex. 17 by letting $x = 1$ and $y = 1$.

27. In a problem in exact division, the dividend is $x^3 + 3x^2y + xy^2 - 2y^3$ and the quotient is $x^2 + xy - y^2$. Find the divisor.

28. In a problem in exact division, the dividend is $x^4 - y^4$ and the quotient is $x^3 + x^2y + xy^2 + y^3$. Find the divisor.

29. Show that $3x - 5$ is a factor of $6x^2 - 31x + 35$.

30. Show that $a + b + c$ is a factor of $a^2 - b^2 - 2bc - c^2$.

31. If $2x - 3y + 1$ is a factor of $4x^2 - 4xy - 3y^2 - 2x + 7y - 2$, find the other factor.

32. If $a^2 + 2a - 1$ is a factor of $2a^4 + 3a^3 - 6a^2 - 3a + 2$, find the other factor.

33. In a problem in division, the dividend is $a^3 - 2a^2 + a - 3$, the divisor is $a + 3$, and the quotient is $a^2 - 5a + 16$. Find the remainder without dividing.

34. In a problem in division, the dividend is $x^4 - 2x^3 - x^2 - x - 1$, the divisor is $x^2 + x + 1$, and the remainder is $x - 2$. Find the quotient.

35. In a problem in division, the dividend is $x^5 + 2x^4 - x^3 + 2x^2 - x + 2$, the quotient is $x^2 + 2x - 2$, and the remainder is $3x^2 + 7x - 4$. Find the divisor.

36. In a problem in division, the divisor is $x^2 + 1$, the quotient is $x^2 + 2x + 2$, and the remainder is $-4x - 1$. Find the dividend.

37. Prove Corollaries 1, 2, and 3 of Theorem 15 (Sec. 2.7).

38. Establish Theorem 16 (Sec. 2.7).

39. If one homogeneous polynomial is exactly divisible by another homogeneous polynomial, show that the quotient is also a homogeneous polynomial whose degree is equal to the difference of the degrees of the dividend and divisor.

40. Show that unity bears a relation to the operations of multiplication and division which is analogous to the relation that zero bears to the operations of addition and subtraction.

2.8. NUMBER FIELD

In anticipation of our discussion of factoring in the next article, we now consider an important concept in mathematics, namely, the concept of *number field* for which we have the following

Definition. A set of numbers is said to constitute a *number field* provided that the sum, difference, product, and quotient (division by zero excluded) of any two equal or distinct numbers of the set are also numbers of the set.

The following sets of numbers are examples of number fields:

(1) All rational numbers.
(2) All real numbers.
(3) All complex numbers.

Let us now consider type 3 of the special products listed in Sec. 2.6, namely,

$$(a + b)(a - b) = a^2 - b^2.$$

Here, given the two factors $a + b$ and $a - b$, we obtained their product $a^2 - b^2$. Conversely, given the product $a^2 - b^2$, the difference of the squares of two numbers, we obtain its factors $a + b$ and $a - b$, the sum and difference, respectively, of the two numbers. From this we may write, corresponding to the three types of number fields above,

(1) $x^2 - 1 = (x + 1)(x - 1)$.
(2) $x^2 - 2 = x^2 - (\sqrt{2})^2 = (x + \sqrt{2})(x - \sqrt{2})$.
(3) $x^2 + 1 = x^2 - i^2 = (x + i)(x - i)$,
where $i = \sqrt{-1}$ and $i^2 = -1$ (Sec. 1.3).

The question now arises, how far shall we go in factoring? Although factoring is done in each of the three fields above, we shall, in general, confine our factoring to the field of rational numbers. That is, our factors shall be rational integral expressions with rational coefficients. Thus, in factoring $a^2 - b^2$, we shall stop with the two factors noted above and not attempt to go further, and, for example, factor $a - b$ in the form

$$a - b = (\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}).$$

An algebraic expression which is the product of two or more factors in a particular number field is said to be *reducible* in that number field, otherwise *irreducible*. Thus, in our three examples above, (1) is reducible in the field of rational numbers but (2) is not. Also, (3) is irreducible in the field of real numbers.

Number fields are important in that a property or theorem which is true in one field may not be true in another field.

2.9. FACTORING

We have seen that in multiplication the problem is to obtain the product of two or more given expressions called the *factors* of that product. We now study the converse problem of obtaining the factors of a given product.

In accordance with the preceding article, we restrict such factoring to the field of rational numbers.

We consider here the factorization of certain types of polynomials which are useful in later work. Most of these are naturally suggested by the various types of special products discussed in Sec. 2.6.

(1) *Common monomial factor.* If each term of an expression has a common monomial factor, that monomial is a factor of the expression as a direct consequence of the distributive law (Sec. 2.5). In general, in factoring *any* expression, if a common factor is present, such a factor should be removed in the very first step.

Example 1. Factor: (*a*) $2ab^2x^2 - 4ab^2xy + 6ab^2y^2$.
(*b*) $3m^2n^3 + 3m^3n^2 - 6mn$.

SOLUTION. (a) $2ab^2x^2 - 4ab^2xy + 6ab^2y^2 = 2ab^2(x^2 - 2xy + 3y^2)$.
(b) $3m^2n^3 + 3m^3n^2 - 6mn = 3mn(mn^2 + m^2n - 2)$.

(2) *Trinomial that is a perfect square.* Types 1 and 2 of the special products of Sec. 2.6,

$$(a \pm b)^2 = a^2 \pm 2ab + b^2,$$

suggest the form assumed by a trinomial that is the square of the binomial sum or difference of two quantities. This is illustrated in

Example 2. Factor $9x^2 - 12xy + 4y^2$.

SOLUTION. $$9x^2 - 12xy + 4y^2 = (3x)^2 - 2(3x)(2y) + (2y)^2 = (3x - 2y)^2.$$

(3) *Difference of two squares.* Factoring in this case is suggested by type 3 of the special products of Sec. 2.6,

$$(a + b)(a - b) = a^2 - b^2,$$

which tells us that the difference of the squares of two quantities has two factors, one the sum and the other the difference of those two quantities.

Example 3. Factor $4a^4x^6 - 25b^6y^4$.

SOLUTION. $$4a^4x^6 - 25b^6y^4 = (2a^2x^3)^2 - (5b^3y^2)^2 = (2a^2x^3 + 5b^3y^2)(2a^2x^3 - 5b^3y^2).$$

(4) *General trinomial.* Here we consider any trinomial other than a perfect square. The forms of the factors are then suggested by type 5 of the special products of Sec. 2.6,

$$(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd.$$

Assuming that the given trinomial is factorable, our problem is to obtain four numbers a, b, c, and d such that a and c are factors of the coefficient of x^2, b and d are factors of the constant term, and the sum of the cross-products ad and bc is the coefficient of x. These numbers are obtained by trial, and the method is best illustrated by means of an example.

Example 4. Factor $6x^2 - 11x - 10$.

SOLUTION. As a first trial we write factors of 6 and of -10 in two separate columns, thus

$\begin{matrix} 6 \\ 1 \end{matrix} \times \left.\begin{matrix} 5 \\ -2 \end{matrix}\right\}$, and then take the sum of the cross-products: $6(-2) + 1(5) = -7$. Since we wish this sum to be -11, we try a different selection of factors, thus

$\begin{matrix} 3 \\ 2 \end{matrix} \times \left.\begin{matrix} 2 \\ -5 \end{matrix}\right\}$, for which the sum of the cross-products is $3(-5) + 2(2) = -11$, the coefficient of x. Hence, our factors are $3x + 2$ and $2x - 5$.

NOTES 1. If the coefficient of x^2 is unity, as in type 4 of the special products of Sec. 2.6, the process is much simpler, for we must then merely determine two numbers whose sum and product are given.

2. If the factors of a quadratic trinomial are not readily obtained by inspection, they may be found by a method which will be discussed later in connection with the quadratic function.

(5) *Polynomial of four terms.* Some polynomials of four terms may be rearranged and grouped so as to exhibit a common factor. This is illustrated in

Example 5. Factor $12xy + 3y - 8x - 2$.

SOLUTION.
$$\begin{aligned} 12xy + 3y - 8x - 2 &= 3y(4x + 1) - 2(4x + 1) \\ &= (4x + 1)(3y - 2). \end{aligned}$$

(6) *Polynomial that is a perfect cube.* We restrict this type to where the given polynomial is the cube of a binomial. The form of such a polynomial is suggested by types 6 and 7 of the special products of Sec. 2.6,

$$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3.$$

Example 6. Factor $8x^3 - 36x^2y + 54xy^2 - 27y^3$.

SOLUTION. That this polynomial may possibly be a perfect cube is suggested by the fact that the first and last terms are perfect cubes, namely, $(2x)^3$ and $(-3y)^3$. We then attempt to write the given polynomial in the form of the cube above, thus

$$\begin{aligned} 8x^3 - 36x^2y + 54xy^2 - 27y^3 &= (2x)^3 - 3(2x)^2(3y) + 3(2x)(3y)^2 - (3y)^3 \\ &= (2x - 3y)^3. \end{aligned}$$

(7) *Sum and difference of two cubes.* The factors in this case are suggested by types 8 and 9 of the special products of Sec. 2.6,

$$(a \pm b)(a^2 \mp ab + b^2) = a^3 \pm b^3.$$

Example 7. Factor $8x^6 + 27y^3$.

SOLUTION. $8x^6 + 27y^3 = (2x^2)^3 + (3y)^3$

$$= (2x^2 + 3y)([2x^2]^2 - [2x^2][3y] + [3y]^2)$$
$$= (2x^2 + 3y)(4x^4 - 6x^2y + 9y^2).$$

NOTE 3. We shall later prove by means of mathematical induction (Chapter 7) that if n is a positive integer,

$x^n + y^n$ has a factor $x + y$ when n is odd,

$x^n - y^n$ has a factor $x - y$ when n is odd or even,

$x^n - y^n$ has a factor $x + y$ when n is even.

In the examples above, the given expressions are readily seen to belong to one of the type forms. Very often, however, a given expression which does not appear to belong to a particular type may be made so by a transformation of some kind, such as a rearrangement of terms or by the addition and subtraction of a particular term. The process is illustrated in the examples following. Here again, as noted in Sec. 2.6, the ability to recognize basic mathematical forms is required.

Example 8. Factor $a^2 + 2ab + b^2 - 3a - 3b - 4$.

SOLUTION. The first three terms represent $(a + b)^2$, and the next two terms represent $-3(a + b)$. This suggests that we have a general trinomial (type 4) in the quantity $a + b$. Hence we write

$$a^2 + 2ab + b^2 - 3a - 3b - 4 = (a + b)^2 - 3(a + b) - 4$$

By type 4, $= ([a + b] + 1)([a + b] - 4)$

$$= (a + b + 1)(a + b - 4).$$

Example 9. Factor $x^4 + 4x^2 + 16$.

SOLUTION. If the second term were $8x^2$, we would have a perfect square. This suggests adding $4x^2$, which would have to be balanced by subtracting $4x^2$. The transformed expression would then be factorable. Thus,

$$x^4 + 4x^2 + 16 = x^4 + 8x^2 + 16 - 4x^2$$
$$= (x^2 + 4)^2 - (2x)^2$$

By type 3, $= (x^2 + 4 + 2x)(x^2 + 4 - 2x)$.

2.10. LOWEST COMMON MULTIPLE

A polynomial that is exactly divisible by a second polynomial is said to be a *multiple* of that polynomial.

Thus, $x^2 - y^2$ is a multiple of $x + y$.

A polynomial that is a multiple of two or more other polynomials is called a *common multiple* of these polynomials.

Thus, $x^2 - y^2$ is a common multiple of $x + y$ and $x - y$.

Evidently two or more polynomials may have more than one common multiple. That common multiple of two or more polynomials which is of least degree is called their *lowest common multiple*, generally designated by the abbreviation *L.C.M.*

The determination of the L.C.M. follows at once from the definition, that is, the L.C.M. of two or more polynomials is equal to the product of all the different factors of these polynomials provided that each factor is taken to a power equal to the greatest number of times it appears in any one polynomial.

Because we will later have occasion to use the L.C.M. of two or more polynomials, we illustrate the determination by means of an example.

Example. Find the L.C.M. of $x^2 - y^2$, $x^2 + 2xy + y^2$, and $x^3 + y^3$.

SOLUTION. We first write each polynomial in factored form, thus

$$x^2 - y^2 = (x + y)(x - y).$$
$$x^2 + 2xy + y^2 = (x + y)^2.$$
$$x^3 + y^3 = (x + y)(x^2 - xy + y^2).$$

The different factors are $x + y$, $x - y$, and $x^2 - xy + y^2$. The greatest number of times that each factor occurs in any one polynomial is twice for $x + y$ and once each for $x - y$ and $x^2 - xy + y^2$. Hence,

$$\text{L.C.M.} = (x + y)^2(x - y)(x^2 - xy + y^2).$$

NOTE. It is generally convenient to leave the L.C.M. in its factored form.

EXERCISES. GROUP 4

In each of Exs. 1–30, factor the given expression.

1. $2x^3y^2 - 6xy^3$.
2. $16a^4 - 24a^2b + 9b^2$.
3. $8b^2m^2 + 24b^2mn + 18b^2n^2$.
4. $9u^2 - 4v^2$.
5. $x^2 + 2xy + y^2 - a^2$.
6. $a^2 + b^2 - c^2 - 2ab$.
7. $m^2 - b^2 - 2mn + n^2$.
8. $x^2 - x - 20$.
9. $6a^2 + 5a - 6$.
10. $6b^2 + 13b - 28$.
11. $12x^2 - 29x + 15$.
12. $2x^2 + 3xy - 2y^2$.

13. $10m^2 - 13mn - 3n^2$.
14. $2a^2 + ab - 6b^2$.
15. $x^2 + 2xy + y^2 + x + y - 6$.
16. $x^2 - 2xy + y^2 + 6x - 6y + 8$.
17. $x^2 + 3x - 2xy - 6y$.
18. $3ax^2 - 6by + 9ay - 2bx^2$.
19. $4a^2mx + 8a^2nx - 2a^2my - 4a^2ny$.
20. $x^3 + 6x^2y + 12xy^2 + 8y^3$.
21. $8x^3 - 12x^2y + 6xy^2 - y^3$.
22. $8x^3 - 64y^3$.
23. $a^3b^6 + 27c^6d^3$.
24. $a^6 - b^6$.
25. $1 + my - y^2 - my^3$.
26. $x^4 + x^2 + 1$.
27. $x^8 + x^4 + 1$.
28. $a^4 + b^4 - 7a^2b^2$.
29. $4x^2y^2 - (x^2 + y^2 - z^2)^2$.
30. $8 - 8x^2 + x^3 - x^5$.

In each of Exs. 31–39, find the L.C.M. of the given expressions, and leave the result in factored form.

31. $2x^2 + 3x - 2$, $6x^2 - 7x + 2$.
32. $6x^2$, $3xy^2$, $12x^3y$.
33. $a^2 + ab - 2b^2$, $3a^2 + 4ab - 4b^2$.
34. $x^4 - 1$, $x^3 + 1$, $2x^2 + 2$.
35. $x^2 - x - 2$, $x^2 + 4x + 3$, $x^2 + x - 6$.
36. $2x^2 - 4xy + 2ax - 4ay$, $6xy - 12by - 12y^2 + 6bx$, $3xy + 3ab + 3ay + 3bx$.
37. $x^4 - 16$, $x^2 + 5x + 6$, $x^2 + x - 6$.
38. $x - y$, $x^2 - y^2$, $x^3 - y^3$, $x^4 - y^4$.
39. $2m^3 + m^2 - 3m$, $m^2 - n - m + mn$, $2m^2 + 2mn + 3m + 3n$.
40. Show that the method of obtaining the L.C.M. of two or more numbers in arithmetic is the same as the method employed in algebra for obtaining the L.C.M. of two or more polynomials.

2.11. SIMPLE FRACTIONS

A *fraction* is the indicated quotient of two quantities. Thus, if A is the dividend and B is the (nonzero) divisor, the quotient A/B is a fraction where A is called the *numerator* and B the *denominator*.

Operations involving fractions are performed the same way in algebra as in arithmetic. Now, however, we are concerned with algebraic expressions instead of simple numbers and must consider negative as well as positive quantities. Since fractions are due to the operation of division, we will have immediate use for the results of Sec. 2.7. For example, the rule of signs for division is directly applicable to fractions.

A *simple algebraic fraction* is one in which the numerator and denominator are rational integral expressions. Examples of simple fractions are

$$\frac{2}{x+1}, \frac{x-1}{x^2+x+4}, \text{ and } \frac{x^2-2x+2}{x+1}.$$

A simple fraction is said to be *proper* if the degree of the numerator is less than the degree of the denominator and *improper* if the degree of the numerator is equal to or greater than that of the denominator. Thus, $\dfrac{2}{x+1}$ and $\dfrac{x-1}{x^2+x+4}$ are proper fractions while $\dfrac{x^2-2x+2}{x^2+1}$ and $\dfrac{x^2-2x+2}{x+1}$ are improper fractions.

An improper fraction may be written as the sum of a polynomial and a proper fraction. Thus, as shown in illustrative Example 4 of Sec. 2.7,

$$\frac{a^3-3a^2+4a-7}{a^2+a-1} = a - 4 + \frac{9a-11}{a^2+a-1}.$$

The following theorem is fundamental in operating with fractions.

Theorem 18. *The value of a fraction remains unchanged if the numerator and denominator are each multiplied* (*or divided*) *by the same nonzero quantity.*

PROOF. By the definition and property of unity (Sec. 2.7), we have

$$\frac{a}{b} = \frac{a}{b} \cdot 1 = \frac{a}{b} \cdot \frac{c}{c}$$

By Theorem 15 (Sec. 2.7), $\qquad = \dfrac{ac}{bc}.$

Since division by a number is equivalent to multiplication by its reciprocal (Theorem 15, Corollary 3), we have, by the first part of the proof,

$$\frac{a}{b} = \frac{a \cdot \frac{1}{d}}{b \cdot \frac{1}{d}} = \frac{\frac{a}{d}}{\frac{b}{d}}.$$

From Theorem 18 we have

Index Law VI. If $a \neq 0$ and m and n are positive integers such that $m < n$,

$$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}.$$

For, by Theorem 18, a^m/a^n remains unaltered if we divide both numerator and denominator by a^m. Then the numerator becomes $a^m/a^m = 1$ by definition of unity, and the denominator becomes $a^n/a^m = a^{n-m}$ by Index Law V (Sec. 2.7).

We now consider in turn the reduction, addition and subtraction, multiplication and division of fractions.

(*1*) *Reduction.* A fraction is said to be in its *lowest terms*, or *simplified*, when its numerator and denominator have no common factor. Evidently a given fraction may be reduced to its lowest terms by dividing both numerator and denominator by any common factors in accordance with Theorem 18. This process is called the *cancellation* of common factors.

Example 1. Reduce $\dfrac{2x^3 - 2x}{4x^4 - 8x^3 - 12x^2}$ to its lowest terms.

SOLUTION. We first factor both numerator and denominator and then divide out (or cancel) any common factors. Thus,

$$\frac{2x^3 - 2x}{4x^4 - 8x^3 - 12x^2} = \frac{2x(x^2 - 1)}{4x^2(x^2 - 2x - 3)} = \frac{2x(x+1)(x-1)}{(2x)^2(x+1)(x-3)} = \frac{x-1}{2x(x-3)}.$$

(*2*) *Addition and subtraction.* If two fractions have a common denominator, their sum or difference is an immediate consequence of Theorem 17 (Sec. 2.7). That is,

$$\text{(1)} \qquad \frac{a}{m} \pm \frac{b}{m} = \frac{a \pm b}{m}.$$

This process may be extended to the algebraic sum of three or more fractions having a common denominator.

If two fractions do not have a common denominator, they may be transformed into equivalent fractions that do have a common denominator, and may then be combined as above. Thus, if b and d are different, then by Theorem 18,

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad}{bd} \pm \frac{bc}{bd}$$

By (1),

$$= \frac{ad \pm bc}{bd}.$$

For simplicity, in transforming two or more given fractions into equivalent fractions having a common denominator, we use their *least common denominator* (L.C.D.) which is the least common multiple (L.C.M.) of their denominators (Sec. 2.10). We illustrate the process by

Example 2. Find the indicated algebraic sum of the fractions in the expression

$$\frac{x}{(x-1)^2} - \frac{x-3}{x^2-1} + \frac{3}{x+1}.$$

SOLUTION. The L.C.D. is readily found to be $(x-1)^2(x+1)$, by Sec. 2.10. The transformation of each fraction into an equivalent fraction whose denominator is the L.C.D. is effected as follows:

$$\frac{x}{(x-1)^2} = \frac{x(x+1)}{(x-1)^2(x+1)} = \frac{x^2+x}{(x-1)^2(x+1)}.$$

$$\frac{x-3}{x^2-1} = \frac{(x-3)(x-1)}{(x^2-1)(x-1)} = \frac{x^2-4x+3}{(x-1)^2(x+1)}.$$

$$\frac{3}{x+1} = \frac{3(x-1)^2}{(x+1)(x-1)^2} = \frac{3x^2-6x+3}{(x-1)^2(x+1)}.$$

Then,

$$\frac{x}{(x-1)^2} - \frac{x-3}{x^2-1} + \frac{3}{x+1} = \frac{x^2+x-(x^2-4x+3)+3x^2-6x+3}{(x-1)^2(x+1)}$$

$$= \frac{3x^2-x}{(x-1)^2(x+1)}.$$

In actual practice, it will usually be found sufficient to simply write out the last statement.

(3) *Multiplication and division.* The product of two fractions is given by Theorem 15 (Sec. 2.7), which states that

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd},$$

that is, *the product of two fractions is another fraction whose numerator and denominator are, respectively, the product of the numerators and the product of the denominators of the given fractions.*

The problem of obtaining the quotient of two fractions is reduced to that of finding the product of two fractions by using the fact that division by a (nonzero) number is equivalent to multiplication by its reciprocal (Theorem 15, Corollary 3). It therefore remains to determine the form of the reciprocal of a fraction. Let r represent the reciprocal of the fraction a/b. Then, since the product of any nonzero number and its reciprocal is equal to unity (Sec. 2.7), we have

$$\frac{a}{b} \cdot r = 1.$$

To this relation we apply in turn the equality laws for multiplication and division. Thus,

Multiplying by b, $\qquad a \cdot r = b,$

Dividing by a, $\qquad r = \dfrac{b}{a}.$

That is, *the reciprocal of a fraction is another fraction with the numerator and denominator interchanged.* The reciprocal of a fraction is said to be obtained by *inverting* the given fraction.

Hence, *the quotient of two fractions is equal to the product of the dividend and the reciprocal of the divisor*, that is,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.$$

Example 3. Divide $\dfrac{x^2 + x - 6}{x^2 - 1}$ by $\dfrac{x^2 - 4}{x + 1}$.

SOLUTION. As discussed above, we invert the divisor and then proceed as in multiplication. Thus,

$$\frac{x^2 + x - 6}{x^2 - 1} \div \frac{x^2 - 4}{x + 1} = \frac{x^2 + x - 6}{x^2 - 1} \cdot \frac{x + 1}{x^2 - 4}$$

$$= \frac{(x^2 + x - 6)(x + 1)}{(x^2 - 1)(x^2 - 4)}.$$

Since we generally obtain the result in simplest form, we reduce this fraction to lowest terms, as in Example 1 above. Thus, factoring both numerator and denominator, we have

$$\frac{(x + 3)(x - 2)(x + 1)}{(x + 1)(x - 1)(x + 2)(x - 2)} = \frac{x + 3}{(x - 1)(x + 2)}.$$

2.12. COMPLEX FRACTIONS

A *complex fraction* is one that has one or more fractions in either the numerator or in the denominator, or in both. Examples of complex fractions are

$$\frac{\dfrac{x + 2}{x^2 - 1} + \dfrac{3}{x + 1}}{\dfrac{2x - 5}{x^2 + 2x - 3}} \quad \text{and} \quad \frac{\dfrac{x + 2}{2x^2 - 3x - 2}}{1 - \dfrac{4}{2x + 1}}.$$

By *the simplification of a complex fraction* we mean its transformation into an equivalent simple fraction reduced to its lowest terms. Two methods may be used. One is to transform the numerator and denominator into simple fractions (if necessary) and then proceed as in the division of fractions (Sec. 2.11). Another method, and often the simpler one, is to obtain a simple fraction by multiplying the original numerator and denominator by the L.C.D. of all the fractions, in accordance with Theorem 18 (Sec. 2.11). We will illustrate both methods by examples.

Example 1. Simplify $\dfrac{\dfrac{x+2}{x^2-1}+\dfrac{3}{x+1}}{\dfrac{2x-5}{x^2+2x-3}}$.

SOLUTION. We will use the first method of dividing one simple fraction by another. Thus,

$$\frac{\dfrac{x+2}{x^2-1}+\dfrac{3}{x+1}}{\dfrac{2x-5}{x^2+2x-3}} = \frac{\dfrac{x+2}{x^2-1}+\dfrac{3(x-1)}{(x+1)(x-1)}}{\dfrac{2x-5}{(x+3)(x-1)}} = \frac{\dfrac{4x-1}{(x+1)(x-1)}}{\dfrac{2x-5}{(x+3)(x-1)}}$$

$$= \frac{4x-1}{(x+1)(x-1)} \cdot \frac{(x+3)(x-1)}{2x-5} = \frac{(4x-1)(x+3)}{(x+1)(2x-5)}.$$

Example 2. Simplify $\dfrac{\dfrac{x+2}{2x^2-3x-2}}{1-\dfrac{4}{2x+1}}$.

SOLUTION. We now illustrate the second method noted above. Since $2x^2 - 3x - 2 = (2x + 1)(x - 2)$, the L.C.D. is $(2x + 1)(x - 2)$. Hence, multiplying numerator and denominator by $(2x + 1)(x - 2)$, we have

$$\frac{\dfrac{x+2}{2x^2-3x-2}}{1-\dfrac{4}{2x+1}} = \frac{x+2}{(2x+1)(x-2)-4(x-2)} = \frac{x+2}{(x-2)(2x-3)}.$$

EXERCISES. GROUP 5

In each of Exs. 1–6, reduce the given fraction to its lowest terms.

1. $\dfrac{a^2+ab}{a^2-b^2}$.
2. $\dfrac{x^2+4x+3}{x^2+2x-3}$.
3. $\dfrac{x^2-y^2}{x^3+y^3}$.
4. $\dfrac{ac-2ad+2bc-4bd}{a^2c+4abc+4b^2c}$.
5. $\dfrac{2x-x^2-x^3}{x^3-3x+2}$.
6. $\dfrac{m^2-mn}{m^3-m^2n+mn-n^2}$.

In each of Exs. 7 and 8, express the given improper fraction as the sum of a polynomial and a proper fraction.

7. $\dfrac{x^3 + 4x^2 - 2x + 1}{x^2 + 1}$.

8. $\dfrac{x^3 + 2}{x + 1}$.

In each of Exs. 9 and 10, transform the given expression into an improper fraction.

9. $x^2 + x + 1 + \dfrac{2}{x - 1}$.

10. $x^2 + 2x + 2 + \dfrac{x + 7}{x^2 - 2}$.

In each of Exs. 11–20, find the indicated algebraic sum.

11. $\dfrac{1}{x} + \dfrac{1}{x + 1} + \dfrac{1}{x - 1}$.

12. $\dfrac{m}{m + 1} - \dfrac{m}{m - 1} + \dfrac{2}{m^2 - 1}$.

13. $\dfrac{1 - x}{2 + x} - \dfrac{1 + x}{2 - x} - \dfrac{3x}{x^2 - 4}$.

14. $\dfrac{2}{a - 1} + \dfrac{a + 1}{a^2 + a + 1} - \dfrac{a^2 - 2}{a^3 - 1}$.

15. $\dfrac{a + 1}{a^2 + a + 1} - \dfrac{2a^3 - 1}{a^4 + a^2 + 1} + \dfrac{a - 1}{a^2 - a + 1}$.

16. $\dfrac{1}{(a - b)(a - c)} - \dfrac{1}{(b - c)(a - b)} - \dfrac{1}{(a - c)(c - b)}$.

17. $\dfrac{a}{(a - b)(a - c)} + \dfrac{b}{(b - c)(b - a)} + \dfrac{c}{(c - a)(c - b)}$.

18. $\dfrac{a^2}{(a - b)(a - c)} + \dfrac{b^2}{(b - c)(b - a)} + \dfrac{c^2}{(a - c)(b - c)}$.

19. $\dfrac{x + y}{(y - z)(z - x)} + \dfrac{y + z}{(z - x)(x - y)} + \dfrac{z + x}{(x - y)(y - z)}$.

20. $\dfrac{b - c}{a^2 - (b - c)^2} + \dfrac{c - a}{b^2 - (c - a)^2} + \dfrac{a - b}{c^2 - (a - b)^2}$.

In each of Exs. 21–28, perform the indicated operation and simplify the result if possible.

21. $\dfrac{5x^2y}{3ab^2} \cdot \dfrac{9a^2b}{10xy^2}$.

22. $\dfrac{(a - 2b)^2}{x^3} \cdot \dfrac{x^2}{a^2 - 4b^2}$.

23. $\dfrac{ax}{a + x} \cdot \left(\dfrac{x}{a} - \dfrac{a}{x}\right)$.

24. $\dfrac{4x^2 - 9y^2}{x^2 - y^2} \div \dfrac{2x + 3y}{x - y}$.

25. $\dfrac{x^2 - 4x + 3}{x^2 + 5x + 6} \div \dfrac{x^2 - 5x + 6}{x^2 + x - 6}$.

26. $\left(a - \dfrac{b^2}{a}\right) \div \left(\dfrac{1}{a} + \dfrac{1}{b}\right)$.

27. $\dfrac{x^3 - 1}{x^3 - 2x^2 - 3x} \cdot \dfrac{x + 1}{x^2 + x - 2} \div \dfrac{x^2 + x + 1}{x^3 - x^2 - 6x}$.

28. $\dfrac{x^2 + xy - xz}{(x + y)^2 - z^2} \div \dfrac{x}{(x + z)^2 - y^2} \cdot \dfrac{xy - y^2 - yz}{(x - y)^2 - z^2}$.

29. Prove that multiplying a fraction by any quantity is equivalent to multiplying its numerator by that quantity.

30. Prove that dividing a fraction by any nonzero quantity is equivalent to multiplying its denominator by that quantity.

In each of Exs. 31–34, evaluate the given complex fraction. Fractions of this type occur in analytic geometry in the determination of the angle between two straight lines.

31. $\dfrac{\frac{4}{3}-\frac{2}{9}}{1+\frac{4}{3}\cdot\frac{2}{9}}$.

32. $\dfrac{\frac{2}{3}+\frac{1}{5}}{1-\frac{2}{3}\cdot\frac{1}{5}}$.

33. $\dfrac{\frac{5}{2}-\frac{3}{7}}{1+\frac{5}{2}\cdot\frac{3}{7}}$.

34. $\dfrac{-\frac{5}{8}-\frac{1}{3}}{1-\frac{5}{8}\cdot\frac{1}{3}}$.

In each of Exs. 35–45, simplify the given complex fraction.

35. $\dfrac{\frac{a^2}{x}-x}{\frac{a}{x}+1}$.

36. $\dfrac{\frac{1}{b}-\frac{1}{a}}{\frac{1}{b^2}-\frac{1}{a^2}}$.

37. $\dfrac{\frac{x}{x-y}-\frac{x}{x+y}}{\frac{y}{x-y}+\frac{x}{x+y}}$.

38. $\dfrac{\frac{m^4-n^4}{m^2-2mn+n^2}}{\frac{m^2+mn}{m-n}}$.

39. $\dfrac{\frac{x^2+x-2}{x^2-x-6}}{\frac{x^2+5x+6}{x^2-9}}$.

40. $1-\dfrac{1}{1+\frac{1}{x}}$.

41. $\dfrac{1}{1-\dfrac{1}{1-\frac{1}{x}}}$.

42. $\dfrac{1}{x+\dfrac{1}{1+\frac{1+x}{1-x}}}$.

43. $\dfrac{x-\frac{x^2+y^2}{y}}{\frac{1}{x}-\frac{1}{y}}\div\dfrac{x^3+y^3}{x^2-y^2}$.

44. $\dfrac{x^2+x+1}{(x+1)^2-x^2}\div\dfrac{x+\frac{1}{1+x}}{1+\frac{x}{1+x}}$.

45. $\left(\dfrac{2}{x}-\dfrac{1}{a+x}+\dfrac{1}{a-x}\right)\div\left(\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}\right)$.

2.13. EXPONENTS

We have already discussed six index laws or laws of exponents (Secs. 2.5, 2.7, 2.11), which are listed here for convenient reference.

I. $a^m \cdot a^n = a^{m+n}$.

II. $(a^m)^n = a^{mn}$.

III. $(ab)^m = a^m b^m$.

IV. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$.

V. $\dfrac{a^m}{a^n} = a^{m-n}, \quad m > n.$

VI. $\dfrac{a^m}{a^n} = \dfrac{1}{a^{n-m}}, \quad m < n.$

It should be particularly noted that these laws have been established only for exponents that are positive integers. On the basis that the laws hold also for exponents that are other than positive integers, we will now determine the significance of fractional, zero, and negative exponents.

Let q be a positive integer so that $1/q$ is a positive fraction. We now consider the meaning of $1/q$ as an exponent, that is, the meaning of $a^{1/q}$ where $a \neq 0$. In order that Index Law I may hold for such a fractional exponent, we must have

$$a^{1/q} \cdot a^{1/q} \cdot a^{1/q} \cdots \text{to } q \text{ factors} = a^{1/q+1/q+1/q+\cdots \text{to } q \text{ terms}}$$
$$= a^{\left(\frac{1}{q}\right)q} = a.$$

That is, $a^{1/q}$ has the property that its qth power is equal to a. We then *define* $a^{1/q}$ *as a* qth *root of* a and write

$$a^{1/q} = \sqrt[q]{a},$$

where the symbol $\sqrt{\ }$ is called the *radical sign* and the integer q is called the *index of the root* (see Sec. 1.3). For $q = 2$, it is customary to omit the index of what we usually call the *square root*.

NOTE. We shall see later that any number (except zero) has q distinct qth roots, and this is the reason for referring to $a^{1/q}$ as "a" qth root of a. Thus, the number 4 has two square roots, $+2$ and -2. In order to avoid this ambigutiy, we shall assign to $a^{1/q}$ a unique value called the *principal root* and defined as follows:

q even. If a is positive, there are two real roots, numerically equal but opposite in sign. The positive root is then taken as the principal root. Thus, the principal square root of 4 is $+2$, denoted by $4^{1/2}$ and the principal fourth root of 81 is $+3$, denoted by $81^{1/4}$.

If a is negative, there are no real roots, and this situation will be considered later (Chapter 8).

q odd. If a is positive, there is one real positive root, and this is taken as the principal root. If a is negative, there is one real negative root, and this is taken as the principal root. Thus, the principal cube root of 8 is +2, denoted by $8^{1/3}$; the principal cube root of −8 is −2, denoted by $(-8)^{1/3}$.

More generally, if p and q are positive integers, then for Index Law II to hold, we must have

$$(a^{p/q})^q = a^{(p/q \cdot q)} = a^p,$$

whence we have the definition

$$a^{p/q} = \sqrt[q]{a^p},$$

that is, by $a^{p/q}$ we mean *the* qth *root of the* pth *power of* a. As before, we restrict this qth root to the principal root. Thus,

$$8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4.$$

We note, furthermore, that by Index Law II, we may also have

$$a^{p/q} = (a^{1/q})^p = (\sqrt[q]{a})^p,$$

that is, we may also interpret $a^{p/q}$ to mean *the* pth *power of the* qth *root of* a. In other words, if we use only the principal root, a number affected by a fractional exponent may be evaluated by taking the power and the root *in either order*.

Thus, for our previous example, we may also have

$$8^{2/3} = (\sqrt[3]{8})^2 = (2)^2 = 4.$$

Hence, *for a fractional exponent, the numerator signifies a power and the denominator a root.*

In order that Index Law I may hold for a zero exponent, we must have, for $m = 0$,

$$a^0 \cdot a^n = a^{0+n} = a^n,$$

whence, by the definitions of division and unity (Sec. 2.7),

$$a^0 = \frac{a^n}{a^n} = 1, \qquad a \neq 0.$$

That is, *any number, other than zero, having the exponent zero, is equal to one. The symbol* 0^0 *is undefined.*

We next consider the meaning of a negative exponent. Let m be a positive integer so that $-m$ is a negative integer. Then, assuming Index Law I to hold, we have

$$a^m \cdot a^{-m} = a^{m-m} = a^0 = 1,$$

whence
$$a^{-m} = \frac{1}{a^m}, \qquad a \neq 0,$$

and
$$a^m = \frac{1}{a^{-m}}, \qquad a \neq 0.$$

That is, *the meaning of a negative exponent is given by the relation*

$$a^{-m} = \frac{1}{a^m}, \qquad a \neq 0.$$

Hence, in a fraction, any factor may be transferred from the numerator to the denominator, and vice versa, provided that the sign of its exponent is changed.

Thus,
$$\frac{a^2b}{x^{-1}y^2} = \frac{xy^{-2}}{a^{-2}b^{-1}} = \frac{a^2bx}{y^2}, \quad \text{and so forth.}$$

We have now given meanings to fractional, zero, and negative exponents, that is, to all *rational* exponents. It may be shown that these meanings are consistent with all six index laws. Later we will consider *irrational* exponents (Chapter 16).

The operations of algebra may now be performed for all rational exponents in exactly the same manner as for exponents which are positive integers. These will be illustrated by several examples.

Many problems involving exponents are problems in simplification. In general, we will consider a given expression to be simplified when it is expressed in its most compact form, a fraction being reduced to its lowest terms, and any fractional exponents being reduced to their lowest terms (Sec. 2.11).

Example 1. Evaluate (a) $(-27)^{2/3}$; (b) $(32)^{2/5}$; (c) $64^{1/3} \cdot 8^{-2/3}$.

SOLUTION. (a) $(-27)^{2/3} = [(-27)^{1/3}]^2 = (-3)^2 = 9.$

(b)
$$(32)^{2/5} = (32^{1/5})^2 = (2)^2 = 4.$$

(c)
$$64^{1/3} \cdot 8^{-2/3} = \frac{64^{1/3}}{8^{2/3}} = \frac{4}{(8^{1/3})^2} = \frac{4}{2^2} = 1.$$

Example 2. Simplify $\left[(8a^6)^{-1/3} \cdot \dfrac{1}{(a^{-2})^{1/2}}\right]^{-1}$.

SOLUTION. $\left[(8a^6)^{-1/3} \cdot \dfrac{1}{(a^{-2})^{1/2}}\right]^{-1} = \left[\dfrac{1}{(2^3a^6)^{1/3}} \cdot \dfrac{1}{a^{-1}}\right]^{-1}$

$$= \left[\frac{1}{2a^2} \cdot a\right]^{-1} = \left[\frac{1}{2a}\right]^{-1} = 2a.$$

Example 3. Multiply $xy^{-1/2} - x^{1/2} + y^{1/2}$ by $x^{1/2} + y^{1/2} + x^{-1/2}y$.

SOLUTION. We proceed here as with integral exponents. (See Example 3 of Sec. 2.5.) The operation then appears as follows:

$$\begin{array}{llll}
xy^{-1/2} - x^{1/2} + y^{1/2} & & & \\
\quad x^{1/2} + y^{1/2} + x^{-1/2}y & & & \\
\hline
x^{3/2}y^{-1/2} & - x + x^{1/2}y^{1/2} & & \\
 & \quad x - x^{1/2}y^{1/2} + y & & \\
 & \quad\quad x^{1/2}y^{1/2} & - y + x^{-1/2}y^{3/2} & \\
\hline
x^{3/2}y^{-1/2} & \quad\quad + x^{1/2}y^{1/2} & \quad + x^{-1/2}y^{3/2}. &
\end{array}$$

Example 4. Express as a complex fraction and simplify

$$\frac{a^{-1}b^{-2} + a^{-2}b^{-1}}{b^{-2} - a^{-2}}.$$

SOLUTION. $\dfrac{a^{-1}b^{-2} + a^{-2}b^{-1}}{b^{-2} - a^{-2}} = \dfrac{\dfrac{1}{ab^2} + \dfrac{1}{a^2b}}{\dfrac{1}{b^2} - \dfrac{1}{a^2}}$

Multiplying numerator and denominator by a^2b^2, $= \dfrac{a + b}{a^2 - b^2} = \dfrac{1}{a - b}$.

EXERCISES. GROUP 6

1. Show that the meaning attached to the zero exponent in Sec. 2.13 is consistent with Index Laws II–VI.

2. Show that the meaning attached to a negative exponent in Sec. 2.13 is consistent with Index Laws II–VI.

In each of Exs. 3–10, evaluate the given expression.

3. $16^{3/4}$. 4. $(-8)^{4/3}$. 5. $25^{-1/2}$. 6. $4^{5/2} \cdot 2^{-3}$.

7. $(1\frac{7}{9})^{-3/2}$. 8. $\dfrac{(0.008)^{1/3}}{5^{-1}}$. 9. $\dfrac{18^{-1/2}}{32^{-1/2}}$. 10. $\dfrac{8^{-2/3} \cdot 16^{-1/4}}{32^{-3/5}}$.

In each of Exs. 11–18, simplify the given expression and write the result with positive exponents.

11. $(2a^2 \div 8a^{-2})^{-1/2}$. 12. $[-8(x^6y^{-6})^{1/2}]^{1/3}$ 13. $[m^{-1}(m\{m^3\}^{1/2})^{1/5}]^{-2}$.

14. $\left[\dfrac{27^{-1}a^{-1}b^2}{(3a^{1/3})^{-3}b^5}\right]^{-1/3}$. 15. $\dfrac{ma^{1/3}}{9nb} \div \left(\dfrac{3n^{1/2}a^{1/3}}{b^{-1/2}}\right)^{-2}$. 16. $\dfrac{x^{-1} + y^{-1}}{x^{-1}y^{-1}}$.

17. $\left(\dfrac{4x^{-2}}{9y^2}\right)^{-1/2} \div \left(\dfrac{8x^3}{27y^3}\right)^{-1/3}$. 18. $\dfrac{(16x^2y^{-3})^{-1/2}}{(3x^{-4}y)^{-1/4}} \cdot \dfrac{(3^{1/2}x^4y^{1/2})^{-1/2}}{(2x^{1/2}y^{3/4})^{-2}}$.

In each of Exs. 19–27, find the indicated product.

19. $(x^{1/2} + y^{1/2})(x^{1/2} - y^{1/2})$. 20. $(x^{1/2} + y^{1/2})^2$.

21. $(x^{1/2} + x^{-1/2})^2$. 22. $(x + x^{-1})(x - x^{-1})$.

23. $(x^{1/3} - y^{1/3})^3$. 24. $(x^{2/3} + y^{2/3})^3$.

25. $(a^{2/3} + a^{1/3}b^{1/3} + b^{2/3})(a^{1/3} - b^{1/3})$. 26. $(a^2 - 1 + a^{-2})(a^2 + 1 + a^{-2})$.

27. $(m^{5/2} - m^2 + m^{3/2} - m + m^{1/2} - m^0)(m^{1/2} + m^0)$.

In each of Exs. 28–32, perform the indicated division and check the result.

28. $(x - y) \div (x^{1/2} - y^{1/2})$. 29. $(x + y) \div (x^{1/3} + y^{1/3})$.

30. $(x^{3n/2} - x^{-3n/2}) \div (x^{n/2} - x^{-n/2})$. 31. $(a^{1/2} - x) \div (a^{1/6} - x^{1/3})$.

32. $(x^{9/5} - x^{7/5} - 8x + 9x^{3/5} - 7x^{1/5} + 6x^{-1/5}) \div (x^{4/5} + 2x^{2/5} - 3)$.

In each of Exs. 33–40, simplify the given expression.

33. $\left[\dfrac{x^{-1/2}y^{-2/3}}{x^{-1/6}y^{-1}} \div \dfrac{x^{-2}y^2}{(xy)^{-3}}\right]^{-3}$. 34. $\left[\left(\dfrac{x^{-1/c}y^{b/c}}{x^{\frac{b+c}{c}}y^{\frac{1}{c}}}\right)^{-1} \div \left(\dfrac{x}{y}\right)^{\frac{b+c}{c}}\right]^c$.

35. $\dfrac{x^{-2} - 2(xy)^{-1} + y^{-2}}{\left(\dfrac{y}{x}\right)^{-2} + xy^{-1} - 2x^0}$. 36. $\dfrac{[(a^m)^{1/r}(a^q)^{1/n}]^{nr}}{[b^{n/q}(b^{r/m})]^{mq}} \div \left[\left(\dfrac{a}{b}\right)^q\right]^r$.

37. $\dfrac{8 - 2x^2 + 2(4 - x^2)^{1/2}}{1 - \dfrac{x^2}{(4 - x^2)^{1/2}} + \dfrac{2}{\left(1 - \dfrac{x^2}{4}\right)^{1/2}}}$.

38. $\left(\dfrac{x^{a+b}}{x^{c-a}}\right)^{\frac{1}{b-c}} \cdot \left(\dfrac{x^{c+a}}{x^{b-c}}\right)^{\frac{1}{a-b}} \cdot \left(\dfrac{x^{b+c}}{x^{a-b}}\right)^{\frac{1}{c-a}}$.

39. $\left[\dfrac{x^{-1} + y^{-1}}{x^{-1} - y^{-1}}\right]^{-1} \div \left[\dfrac{y^{-2} + x^{-2}}{y^{-2} - x^{-2}}\right]^{-1} + (x^{-3} + y^{-3})^0$.

40. $\dfrac{(x - x^{-1})(y - y^{-1})}{xy + (xy)^{-1}} + \dfrac{x^2 + y^2 - (x^{-2} + y^{-2})}{x^2y^2 - (xy)^{-2}}$.

2.14. RADICALS

The expression $\sqrt[q]{a}$, denoting the principal qth root of a, is called a *radical*, and the quantity a under the radical sign is called the *radicand*. The index of the root, q, is also called the *order* of the radical.

In Sec. 2.13 we made the definition

$$a^{1/q} = \sqrt[q]{a},$$

so that radicals may be replaced by expressions with exponents. Hence, operations with radicals may be performed in accordance with the index laws (Sec. 2.13), it being understood that each root involved is the principal root. From these index laws we may obtain the following *laws of radicals:*

I. $\sqrt[m]{a} \cdot \sqrt[m]{b} = \sqrt[m]{ab}.$

II. $\dfrac{\sqrt[m]{a}}{\sqrt[m]{b}} = \sqrt[m]{\dfrac{a}{b}}, \quad b \neq 0.$

III. $\sqrt[n]{\sqrt[m]{a}} = \sqrt[mn]{a} = \sqrt[m]{\sqrt[n]{a}}.$

We use these laws to simplify radicals and to perform the various algebraic operations on them. It should be noted that when m and n are even, the radicands a and b in these laws are non-negative numbers.

(1) *Simplification.* The single radical $\sqrt[q]{a}$ is said to be in *simplest form* when it satisfies the following conditions:

(a) The radicand contains no factor to a power as high as the order q of the radical.
(b) The radicand contains no fractions.
(c) The index of the radical is as small as possible.

We illustrate the simplification of single radicals in the following example.

Example 1. Simplify: (a) $\sqrt[3]{8a^5}$; (b) $\sqrt{\dfrac{27}{2}}$; (c) $\sqrt[6]{27}$.

SOLUTION. (a) $\sqrt[3]{8a^5} = \sqrt[3]{2^3a^3 \cdot a^2}$

By Law I, $\qquad = \sqrt[3]{2^3a^3} \cdot \sqrt[3]{a^2}$

$$= 2a\sqrt[3]{a^2}.$$

(b) By Law II, $\quad \sqrt{\dfrac{27}{2}} = \dfrac{\sqrt{27}}{\sqrt{2}} = \dfrac{\sqrt{3^2 \cdot 3}}{\sqrt{2}}$

By Law I, $\qquad = \dfrac{3\sqrt{3}}{\sqrt{2}}.$

The next step is the removal of the radical from the denominator, a process known as *rationalizing the denominator.* It is accomplished by multiplying both numerator and denominator by $\sqrt{2}$. Thus,

$$\frac{3\sqrt{3}}{\sqrt{2}} = 3 \cdot \frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

By Law I,
$$= \frac{3\sqrt{3 \cdot 2}}{2} = \frac{3}{2}\sqrt{6}.$$

This result may have been obtained more directly as follows:

$$\sqrt{\frac{27}{2}} = \frac{\sqrt{27}}{\sqrt{2}} = \frac{\sqrt{27}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{54}}{2} = \frac{\sqrt{9 \cdot 6}}{2} = \frac{3}{2}\sqrt{6}.$$

On account of its great importance, this process will be considered in greater detail under Item (4).

(c) $$\sqrt[6]{27} = \sqrt[6]{3^3} = 3^{3/6} = 3^{1/2} = \sqrt{3}.$$

(2) *Addition and subtraction.* Two radicals are said to be *similar* if, when reduced to their simplest forms, they have the same radicand and the same index.

Thus, $3\sqrt[4]{7}$ and $-2\sqrt[4]{7}$ are similar radicals.

The algebraic sum of similar radicals, regarded as similar terms, is obtained by multiplying the sum of their coefficients by their common radical factor.

Example 2. Find the indicated sum:

$$4\sqrt{2} - 2\sqrt{18} + 3\sqrt{32} - \sqrt{50}.$$

SOLUTION. If possible, we first simplify any terms. Thus,

$$4\sqrt{2} - 2\sqrt{18} + 3\sqrt{32} - \sqrt{50} = 4\sqrt{2} - 2\sqrt{9 \cdot 2} + 3\sqrt{16 \cdot 2} - \sqrt{25 \cdot 2}$$

$$= 4\sqrt{2} - 6\sqrt{2} + 12\sqrt{2} - 5\sqrt{2} = 5\sqrt{2}.$$

(3) *Multiplication and division.* To multiply two radicals, we first transform them, if necessary, to the same order and then apply Law I. The procedure is illustrated in

Example 3. Multiply $\sqrt[3]{2}$ by $\sqrt{3}$.

SOLUTION. The indices 3 and 2 have 6 as their L.C.M. We therefore transform each radical to the order 6. Thus,

$$\sqrt[3]{2} = 2^{1/3} = 2^{2/6} = \sqrt[6]{4}.$$

$$\sqrt{3} = 3^{1/2} = 3^{3/6} = \sqrt[6]{27}.$$

Hence, $\sqrt[3]{2} \cdot \sqrt{3} = \sqrt[6]{4} \cdot \sqrt[6]{27}$

By Law I, $= \sqrt[6]{4 \cdot 27} = \sqrt[6]{108}.$

The multiplication of expressions containing two or more terms, some or all of which are radicals, is performed as for ordinary algebraic expressions (Sec. 2.5). In this process, the multiplication of individual radicals is carried out as above.

Example 4. Multiply $3\sqrt{x} + 2\sqrt{y}$ by $2\sqrt{x} - 3\sqrt{y}$.

SOLUTION. We arrange the expressions and proceed as in ordinary multiplication. The work then appears as follows:

$$\begin{array}{l} 3\sqrt{x} + 2\sqrt{y} \\ \underline{2\sqrt{x} - 3\sqrt{y}} \\ 6x + 4\sqrt{xy} \\ \qquad \underline{-9\sqrt{xy} - 6y} \\ 6x - 5\sqrt{xy} - 6y. \end{array}$$

To divide one radical by another we first transform them, if necessary, to the same order, and then apply Law II.

Example 5. Perform the indicated divisions:

$$\text{(a)}\ \frac{\sqrt{10}}{\sqrt{2}}; \qquad \text{(b)}\ \frac{\sqrt{7}}{\sqrt{2}}; \qquad \text{(c)}\ \frac{\sqrt{3}}{\sqrt[3]{3}}.$$

SOLUTION. (a) By Law II, $\dfrac{\sqrt{10}}{\sqrt{2}} = \sqrt{\dfrac{10}{2}} = \sqrt{5}.$

(b) In this case, if we apply Law II directly, we obtain $\sqrt{\dfrac{7}{2}}$, which is not in its simplest form. Hence, we proceed as in Example 1(b) and rationalize the denominator. Thus,

$$\frac{\sqrt{7}}{\sqrt{2}} = \frac{\sqrt{7} \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} = \frac{\sqrt{14}}{2}.$$

(c) Transforming both radicals to the same order 6, we have

$$\frac{\sqrt{3}}{\sqrt[3]{3}} = \frac{\sqrt[6]{27}}{\sqrt[6]{9}} = \sqrt[6]{3}.$$

If the dividend consists of several terms and the divisor is a single radical, the operation of division is performed by dividing each term of the dividend by the divisor, as illustrated in the preceding example. However, if the divisor consists of two or more terms of which at least one is a radical, it is generally desirable to rationalize the divisor. This operation is discussed next.

(4) *Rationalization of the denominator.* If we desire to evaluate $1/\sqrt{2}$ as it stands, using a value of 1.414 for $\sqrt{2}$, we must divide unity by 1.414. However, if we first rationalize the denominator, as in Examples 1(b) and 5(b), the evaluation is much simpler. Thus,

$$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1.414}{2} = 0.707.$$

In general, by the *rationalization of the denominator* of a given fraction, we mean the transformation of that fraction into an equivalent expression whose denominator is rational. We now extend this process to the case where the denominator of a fraction is a radical expression consisting of two or more terms.

One radical expression is said to be a *rationalizing factor* for another radical expression if their product is rational. Thus, $\sqrt{a} - \sqrt{b}$ and $\sqrt{a} + \sqrt{b}$ are rationalizing factors for each other, for

$$(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) = a - b.$$

We illustrate the use of a rationalizing factor in the following example.

Example 6. Divide $\sqrt{22}$ by $2\sqrt{3} + \sqrt{11}$.

SOLUTION. In view of our previous discussion, we consider the equivalent problem of rationalizing the denominator of the fraction

$$\frac{\sqrt{22}}{2\sqrt{3} + \sqrt{11}}.$$

A rationalizing factor for the denominator is obviously $2\sqrt{3} - \sqrt{11}$. Hence, we have

$$\frac{\sqrt{22}}{2\sqrt{3} + \sqrt{11}} = \frac{\sqrt{22}}{2\sqrt{3} + \sqrt{11}} \cdot \frac{2\sqrt{3} - \sqrt{11}}{2\sqrt{3} - \sqrt{11}}$$
$$= \frac{2\sqrt{66} - 11\sqrt{2}}{12 - 11} = 2\sqrt{66} - 11\sqrt{2}.$$

Sometimes the rationalization process may require more than one step. This is illustrated in

Example 7. Rationalize the denominator of $\dfrac{1}{1 + \sqrt{2} - \sqrt{3}}$.

SOLUTION. Since a complete rationalizing factor is not obvious, we multiply numerator and denominator by $1 + \sqrt{2} + \sqrt{3}$ as a first step.

Thus,

$$\frac{1}{1+\sqrt{2}-\sqrt{3}} = \frac{1}{(1+\sqrt{2})-\sqrt{3}} \cdot \frac{1+\sqrt{2}+\sqrt{3}}{(1+\sqrt{2})+\sqrt{3}} = \frac{1+\sqrt{2}+\sqrt{3}}{(1+\sqrt{2})^2-3}$$
$$= \frac{1+\sqrt{2}+\sqrt{3}}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}+2+\sqrt{6}}{4}.$$

EXERCISES. GROUP 7

1. By means of the index laws of Sec. 2.13, establish the laws of radicals given in Sec. 2.14.

2. If m and n are positive integers, show that
$$\sqrt[n]{a^m} = (\sqrt[n]{a})^m.$$

3. If m, n, and p are positive integers, show that
$$\sqrt[n]{a^m} = \sqrt[np]{a^{mp}}.$$

In each of Exs. 4–11, reduce the given radical to its simplest form.

4. $\sqrt{8a^3}$. 5. $\sqrt[3]{-27b^5}$. 6. $\sqrt[4]{32m^5n^6}$. 7. $\sqrt{45a^5x^3}$.

8. $\sqrt{\frac{5}{3}}$. 9. $\sqrt{\frac{75a^3}{2}}$. 10. $\sqrt[4]{25m^2}$. 11. $\sqrt[6]{8a^3}$.

In each of Exs. 12–15, find the indicated sum.

12. $\sqrt{20} - \sqrt{45} + \sqrt[4]{25} + 2\sqrt[6]{125}$.

13. $\sqrt{48} - 2\sqrt[4]{9} + \sqrt{8} - \sqrt{50}$.

14. $\sqrt{28} + 21\sqrt{\frac{1}{7}} - \frac{4\sqrt{14}}{\sqrt[4]{4}} + \sqrt[4]{49}$.

15. $\sqrt[3]{48} + 2\sqrt[6]{4} - 3\sqrt[3]{6} - \sqrt[3]{2} + 2\sqrt[6]{36}$.

In each of Exs. 16–31, perform the indicated operation.

16. $(2\sqrt{15})(\sqrt{27})$.

17. $(3\sqrt{6})(\sqrt{8})$.

18. $(\sqrt[3]{3})(\sqrt{2})$.

19. $(\sqrt{15}) \div (\sqrt{3})$.

20. $(\sqrt{15}) \div (\sqrt{6})$.

21. $(\sqrt[3]{4}) \div (\sqrt[4]{2})$.

22. $(3\sqrt{2} + 2\sqrt{3} - \sqrt{5})\sqrt{3}$.

23. $(3\sqrt{2} + 2\sqrt{3} - \sqrt{5}) \div \sqrt{3}$.

24. $(\sqrt[3]{4} - 2\sqrt{3} - \sqrt{2})\sqrt[3]{2}$.

25. $(\sqrt[3]{4} - 2\sqrt{3} - \sqrt{2}) \div \sqrt[3]{2}$.

26. $(2\sqrt{x} - 3\sqrt{y})^2$.

27. $(3\sqrt{2} - 2\sqrt{3})^2$.

28. $\sqrt{7+\sqrt{13}} \cdot \sqrt{7-\sqrt{13}}$.

29. $(\sqrt{3} - \sqrt{2})^3$.

30. $\sqrt[3]{7+\sqrt{22}} \cdot \sqrt[3]{7-\sqrt{22}}$.

31. $(\sqrt{2} + \sqrt{3} + \sqrt{5})(\sqrt{2} + \sqrt{3} - \sqrt{5})$.

32. Find the value of $x^2 + 2x - 2$ when $x = -1 - \sqrt{3}$.

33. Find the value of $2x^2 + 3x - 3$ when $x = \dfrac{-3 + \sqrt{33}}{4}$.

In each of Exs. 34–43, rationalize the denominator.

34. $\dfrac{3}{\sqrt{3} - 3}$. 35. $\dfrac{2\sqrt{3}}{\sqrt{3} - \sqrt{2}}$. 36. $\dfrac{2 - \sqrt{6}}{5 + 2\sqrt{6}}$.

37. $\dfrac{\sqrt{3} + \sqrt{2}}{\sqrt{3} - \sqrt{2}}$. 38. $\dfrac{1}{2 + \sqrt{3} + \sqrt{5}}$. 39. $\dfrac{1}{\sqrt{2} + \sqrt{3} + \sqrt{5}}$.

40. $\dfrac{\sqrt{a + 1} + \sqrt{a}}{\sqrt{a + 1} - \sqrt{a}}$. 41. $\dfrac{\sqrt{1 + a^2} - \sqrt{1 - a^2}}{\sqrt{1 + a^2} + \sqrt{1 - a^2}}$.

42. $\dfrac{\sqrt{2} - \sqrt{3} - \sqrt{5}}{\sqrt{2} + \sqrt{3} + \sqrt{5}}$. 43. $\dfrac{\sqrt{2} - \sqrt{5} - \sqrt{7}}{\sqrt{2} + \sqrt{5} + \sqrt{7}}$.

44. Simplify $\dfrac{8 - 2x^2 + 2\sqrt{4 - x^2}}{1 - \dfrac{x^2}{\sqrt{4 - x^2}} + \dfrac{2}{\sqrt{1 - \dfrac{x^2}{4}}}}$.

45. Simplify $\dfrac{\sqrt{a}}{\dfrac{\sqrt{(a - b)^3}}{(x + 1)^2} - \dfrac{a - b}{x + 1}} - \dfrac{\sqrt{a^2 - ab}}{\dfrac{\sqrt{(a - b)^3(x - 1)}}{x^2 - 1} - a + b}$.

2.15. A NECESSARY AND SUFFICIENT CONDITION

Now we shall consider the meaning of the expression "a necessary and sufficient condition," which occurs frequently in mathematics. We will first illustrate its meaning by an example. The student is already familiar with the following simple theorem from elementary geometry:

If a triangle is isosceles, the angles opposite the equal sides are equal.

This theorem states that if a triangle is isosceles, it *necessarily* follows that the angles opposite the equal sides are equal. Hence we say that the existence of two equal angles is a *necessary condition* that the triangle be isosceles.

But the *converse* of this theorem is also true, namely,

If two angles of a triangle are equal, the sides opposite these angles are equal, that is, the triangle is isosceles.

This theorem states that the existence of two equal angles is *sufficient* to make the triangle isosceles. Hence we say that the existence of two equal angles is a *sufficient condition* that the triangle be isosceles. We may then combine both the theorem and its converse in the following single statement: *A necessary and sufficient condition* that a triangle be isosceles is that two of its angles are equal.

An alternative phrase often used in place of a necessary and sufficient condition is "*if and only if*." Thus, the preceding statement may be written: A triangle is isosceles *if and only if* two of its angles are equal.

More generally, if the hypothesis A of a theorem implies the truth of a conclusion B, then B is a *necessary condition* for A. Furthermore, if, conversely, B implies the truth of A, then B is a *sufficient condition* for A.

In Sec. 2.5, we established Theorem 8 and its converse, Theorem 11, which are restated here for convenience:

Theorem 8. *The product of any number and zero is equal to zero.*

Theorem 11. *If the product of two numbers is equal to zero, at least one of these numbers is equal to zero.*

Now, in view of our previous discussion, we may combine these two theorems in the following single statement: *A necessary and sufficient condition* that the product of two numbers be equal to zero is that at least one of these numbers is equal to zero.

A generalization of Theorem 11 was stated in a corollary to the theorem. Hence, in view of the great importance of this theorem in the solution of equations, we restate the previous theorem in the following form:

Theorem 19. *The product of two or more factors is equal to zero if and only if at least one of these factors is equal to zero.*

Later we shall often have occasion to refer to this theorem.

We next consider the concept of a necessary and sufficient condition in connection with the meaning of the term *definition*. By the *definition of an object* is meant a description of that object of such a nature that it is possible to identify it definitely among all other objects of its class. The implication of this statement should be carefully noted: it expresses a *necessary and sufficient condition* for the existence of the object defined. Thus, consider that we are defining an algebraic expression of type A by means of a unique property P which A possesses. Then, in the entire class of all algebraic expressions, an expression is of type A *if and only if* it possesses property P.

As a specific example, let us consider the term *rational* number, which was defined in Sec. 1.3 as the number having the unique property P that it can be expressed in the form p/q where p is any positive or negative integer or zero, and q is any positive or negative integer. This means that every rational number has property P; and conversely, any number having property P is a rational number. To emphasize this characteristic, we may reword our definition thus: A number is rational *if and only if* it can be expressed in the form p/q where p is any positive or negative integer or zero, and q is any positive or negative integer.

As we proceed in our study of algebra we will often have occasion to derive various necessary and sufficient conditions.

2.16. SUMMARY

In this chapter we studied all six algebraic operations applied to real numbers and to various algebraic expressions representing real numbers. We have not, however, considered complex numbers, since, as previously indicated, we shall make a special study of such numbers in a later chapter.

Further work in algebra is concerned with various topics and applications in which algebraic operations are used. The student should not hesitate, therefore, to refer back to this chapter whenever he feels the need for refreshing his memory on the proper procedure in algebraic operations.

We conclude this chapter with a group of miscellaneous exercises which, in general, are somewhat more difficult than those in the previous groups. The student will find some of these exercises intriguing and challenging.

EXERCISES. GROUP 8

1. If $a > b$ and $b > c$, show that $a > c$.

2. If a and b are two different numbers, show that there is a number $x = \dfrac{a+b}{2}$, which lies between them. That is, if $a > b$, show that $a > x > b$.

3. In Ex. 2, if $a < b$, show that $a < x < b$.

4. If $\dfrac{a}{b} = \dfrac{c}{d}$, show that $\dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}$.

5. If $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3} = r$, show that $\dfrac{a_1 + a_2 + a_3}{b_1 + b_2 + b_3} = r$.

6. Find the fallacy in the following demonstration:

Let $a = b$.

Multiplying by a,	$a^2 = ab.$
Subtracting b^2,	$a^2 - b^2 = ab - b^2.$
Factoring,	$(a+b)(a-b) = b(a-b).$
Dividing by $a - b$,	$a + b = b.$
Since $a = b$,	$b + b = b,$
or	$2b = b,$
whence	$2 = 1.$

7. In the ordinary arithmetic multiplication of 47 by 32, show how the distributive law is used.

8. If $s = a + b + c$, show that $s(s-2a)(s-2b) + s(s-2b)(s-2c) + s(s-2c)(s-2a) = (s-2a)(s-2b)(s-2c) + 8abc$.

9. The algebraic operations of addition, subtraction, multiplication, division, and involution are called the *rational operations*. Justify the use of this term by showing that if one or more of these operations are performed on rational numbers, the results are also rational numbers.

10. Factor $2a^2 - b^2 + ab - 3a + 3b - 2$.

11. Factor $3x^2 - 5xy - 2y^2 + 7x + 7y - 6$.

12. Factor $a^4 + 4$.

13. If a is a positive integer greater than 1, show that $a^3 - a$ is exactly divisible by 6.

14. Find the L.C.M. of $x^2 + x - 2$, $x^3 - 13x + 12$, and $x^3 + 3x^2 - 10x - 24$.

15. *The highest common factor* (H.C.F.) of two or more polynomials is the polynomial of highest degree which is an exact divisor of each of them. Find the H.C.F. and L.C.M. of $ax^2 - ay^2$ and $ax^2 + axy - 2ay^2$.

16. Let H represent the H.C.F. and L the L.C.M. of any two polynomials P and Q. Prove that $H \times L = P \times Q$. Verify this theorem for Ex. 15.

In Exs. 17–19, p, q, r and s are positive integers.

17. Show that $(a^{p/q})(a^{r/s}) = a^{p/q+r/s}$.

18. Show that $(a^{p/q})^{r/s} = a^{pr/qs}$.

19. Show that $\dfrac{a^{p/q}}{a^{r/s}} = a^{p/q-r/s}$.

20. If we do not restrict the root of a number to its principal root, show, by means of an example, that the pth power of a qth root of a number is not always equal to a qth root of its pth power.

21. Without using a table of roots, determine which is the greater:

(a) $\sqrt{5}$ or $\sqrt[3]{11}$? (b) $\sqrt[3]{14}$ or $\sqrt{6}$?

22. If the correct value of $\sqrt{2}$ to 7 decimal places is 1.4142136, find the value of $1/(\sqrt{2} - 1)$ correct to 7 decimal places.

23. If the correct value of $\sqrt{3}$ to 7 decimal places is 1.7320508, find the value of $1/(2 - \sqrt{3})$ correct to 7 decimal places.

24. Show that $\sqrt{2}$ is irrational by the following procedure. Assume, contrary to the desired result, that $\sqrt{2}$ is rational so that we may write the equation $\sqrt{2} = a/b$, where a and b are integers and have no integral factor in common. Then show that this equation leads to a contradiction.

25. Show that $\sqrt{3}$ is irrational.

26. Rationalize the denominator of $\dfrac{1}{1 - \sqrt{2} + \sqrt{3}}$, and hence determine the rationalizing factor which would have given the result in one step.

27. Rationalize the denominator of $\dfrac{1}{1 - \sqrt[3]{a}}$.

28. Rationalize the denominator of $\dfrac{1}{\sqrt[3]{x} + \sqrt[3]{y}}$.

29. Rationalize the denominator of $\dfrac{1}{\sqrt[3]{3} - \sqrt[3]{2}}$.

30. Find a rationalizing factor for $\sqrt{x} - \sqrt[3]{y}$.

31. Find the positive square root of $29 + 12\sqrt{5}$ as a radical expression in simplest form.

32. Find the positive square root of $5 + 2\sqrt{6}$ as a radical expression in simplest form.

33. If a and b are positive numbers, explain the fallacy in the statement that $\sqrt{-a} \cdot \sqrt{-b} = \sqrt{ab}$. Give the correct statement.

34. Show, by means of examples, that a condition may be necessary without being sufficient, and vice versa.

35. Show, by means of examples, that there may be more than one necessary and sufficient condition for the truth of a theorem.

3

The function concept

3.1. INTRODUCTION

This chapter will consider the meaning of the term *function*, a concept of fundamental importance in mathematics. Our treatment of function at this time is basic and general, but the student will observe the development of this concept as we proceed further here and in succeeding chapters.

3.2. CONSTANTS AND VARIABLES

In a given expression or relation, or in the course of a particular discussion, there may exist two types of quantities for which we have the following

Definitions. A symbol which represents a fixed value is called a *constant*; a symbol which may represent different values is called a *variable*. The set of values which a variable may assume is called the *range* of the variable.

For example, consider the formula $C = 2\pi r$, which gives the circumference C of a circle of radius r. In this relation, C and r may assume various (related) values and hence are variables, but the quantities 2 and π are both fixed and are therefore constants.

There are two types of constants, absolute and arbitrary. An *absolute constant* is one which has the same value in every problem or situation. Thus 2 and π are absolute constants. An *arbitrary constant* or *parameter* is one which retains the same value throughout a particular problem or situation, but this value may be different in another problem or situation. For example, consider the expression $ax + b$, where x may take on different values but a and b are considered constants. This is a general expression or formula for polynomials of the first degree, such as $2x + 5$ where

$a = 2$ and $b = 5$, and $x - 4$ where $a = 1$ and $b = -4$. In this case, a and b are arbitrary constants or parameters.

3.3. DEFINITION OF FUNCTION

We first define a function of a single variable as follows:

If two variables x and y are so related that for each permissible value assigned to x within its range there corresponds one or more values of y, then y is said to be a function of x.

For example, the relation $y = 2x + 5$ exhibits y as a function of x since, for each value assigned to x, a corresponding value of y is determined. For this particular function the student may easily verify several pairs of corresponding values as given in the following table:

x	0	1	2	−1	−2	−3
y	5	7	9	3	1	−1

We note that we may assign values to x at pleasure, but the resulting values of y *depend* upon the values assigned to x in a particular function. For this reason, we call x the *independent variable* and y the *dependent variable*.

The student will observe that the function concept is concerned with the *dependence* of one quantity upon another. Such a relation occurs in a wide variety of cases. Thus, in the formula previously cited, $C = 2\pi r$, the circumference C of a circle is a function of its radius r, that is, the circumference of a circle depends upon its radius.

In our definition of function we referred to each *permissible value* assigned to x. The reason for using the word permissible is that in a given functional relation the independent variable may not necessarily assume all values. For example, in the function $x/(x - 1)$, x may assume all values except 1, for division by zero is an excluded operation (Sec. 2.7). Also, in the relation $y = \sqrt{x}$, if we restrict y to real values, we cannot assign negative values to x.

A function of x is said to be *defined* for a particular value of x provided it has a definite value for that value of x. Thus, as indicated above, the function $x/(x - 1)$ is not defined for $x = 1$. Also, for real values, the function $\sqrt{x}$ is defined only for non-negative values of x.

3.4. TYPES OF FUNCTIONS

If a function has one and only one value for each value assigned to the independent variable, it is called a *single-valued function;* if it has more than one, it is called a *multiple-valued function*. Thus in the relation $y = 2x + 5$, y is a single-valued function of x because, for each value assigned to x, one and only one value of y is determined. But in the relation $y = \pm\sqrt{x + 1}$, y is a double-valued function of x since, for each value assigned to x, two corresponding values of y are determined.

If the variable y is expressed directly in terms of the variable x, it is said to be an *explicit function* of x. Thus in the relation $y = 2x + 5$, y is an explicit function of x. If, however, the variables x and y appear in a relation where neither variable is expressed directly in terms of the other, then either variable is said to be an *implicit function* of the other. Thus in the relation $x + y = 5$, y is an implicit function of x and x is an implicit function of y.

Consider now that x and y are connected by a relation such that y is an explicit function of x. If this relation is transformed so that x is expressed as an explicit function of y, then this latter function of y is called the *inverse function* of the original function of x. Thus, the function $y = 5 - x$ is readily transformed into its inverse function $x = 5 - y$.

Another distinction between various types of functions is the *number of independent variables* involved. In Sec. 3.3 the definition of function was restricted to a *single* independent variable. We may, however, have functions of two or more variables. For example, in the relation $z = x^2 - y^2$, the dependent variable z is a function of the two independent variables, x and y. Here we may assign values to x and y independently of each other. Functions of this type are called *functions of several variables*. As for functions of a single variable, we may have single-valued, multiple-valued, explicit, implicit, and inverse functions of several variables.

3.5. FUNCTIONAL NOTATION

Heretofore we have, for convenience, let the letter y represent a function of x; thus, $y = 2x + 5$. We may, however, also use the symbol $f(x)$ in place of y and write

(1) $$y = f(x) = 2x + 5,$$

where $f(x)$ is read " the f function of x" or simply, "f of x." But this symbol has another very important use. If we wish to express the value of this function when the independent variable x has a particular value, say a, we merely replace x by a wherever it occurs in the function. Hence, for the function given by relation (1), we have

$$f(a) = 2a + 5.$$

Similarly, for the same function we have

$$f(0) = 2(0) + 5 = 5,$$
$$f(-1) = 2(-1) + 5 = 3, \text{ and so on.}$$

In a particular problem, $f(x)$ represents a definite function. But if more than one function occurs in the same discussion, we use different letters to distinguish them, such as $F(x)$, $g(x)$, and $\phi(x)$. For example, to distinguish another function of x from (1), we could write

$$F(x) = x^2 + x - 1.$$

We may also extend this same symbolism or functional notation to functions of several variables. Thus, if $z = x^2 - xy + 2y^2$, we may write

$$z = f(x, y) = x^2 - xy + 2y^2,$$

whence

$$f(a, b) = a^2 - ab + 2b^2,$$
$$f(y, x) = y^2 - yx + 2x^2,$$
$$f(2, 3) = 2^2 - (2)(3) + 2(3)^2 = 16, \text{ and so on.}$$

Furthermore, in accordance with this functional notation, if y is an explicit function of x, we may write $y = f(x)$ from which we may obtain its inverse function and write it symbolically in the form $x = g(y)$. Also, if x and y are implicit functions of each other, as in the relation $x + y - 5 = 0$, we may indicate this by the notation $F(x, y) = 0$.

Example 1. If $f(x) = \dfrac{x+1}{x-1}$ and $F(x) = \dfrac{x}{x+1}$, find $\dfrac{f(2) + F(1)}{1 - f(2) \cdot F(1)}$.

SOLUTION. In accordance with the significance of functional notation, we have.

$$\frac{f(2) + F(1)}{1 - f(2) \cdot F(1)} = \frac{\dfrac{2+1}{2-1} + \dfrac{1}{1+1}}{1 - \dfrac{2+1}{2-1} \cdot \dfrac{1}{1+1}} = \frac{3 + \frac{1}{2}}{1 - \frac{3}{2}} = \frac{6+1}{2-3} = -7.$$

Example 2. If $f(y) = \dfrac{y}{y^2 - 1}$ and $g(y) = \dfrac{1}{y+1}$, find $f[g(y)]$.

SOLUTION. The expression $f[g(y)]$ is often called a function of a function. It means that each y in the expression for $f(y)$ is to be replaced by the expression for $g(y)$. Thus,

$$f[g(y)] = \frac{\dfrac{1}{y+1}}{\dfrac{1}{(y+1)^2} - 1} = \frac{y+1}{1 - (y+1)^2} = \frac{y+1}{-y^2 - 2y}.$$

EXERCISES. GROUP 9

1. The volume V of a right circular cone of base radius r and altitude h is given by the formula $V = \frac{1}{3}\pi r^2 h$. Express (a) the altitude h as an explicit function of V and r and (b) the radius r as an explicit function of V and h.

2. The period of vibration T of a pendulum of length L is given by the formula $T = 2\pi\sqrt{\frac{L}{g}}$, where g is the constant acceleration of gravity. Express L as a function of T.

3. Express the length d of the diagonal of a square as a function of its area A.

4. For a circle of radius r, the circumference C is given by the formula $C = 2\pi r$ and the area A by the formula $A = \pi r^2$. Express the area as a function of the circumference.

5. If $f(x) = x^2 - x + 1$, find $f(1), f(-2), f\left(\frac{2}{3}\right)$.

6. If $f(x) = x^4 - 5x^2 + 4$, find $f(1), f(-1), f(2), f(-2)$.

7. If $f(x) = x + \frac{1}{x}$, show that $f(t) = f\left(\frac{1}{t}\right)$ and that $f(-t) = -f(t)$.

8. If $g(x) = x^6 + x^4 - x^2 + 2$, show that $g(-x) = g(x)$.

9. If $\phi(y) = \sqrt{y^2 + 9}$, find $\phi(\sqrt{7}), \phi(4), \phi(0)$.

10. If $F(x) = x^2 - 3x + 1$, find $F\left(\frac{3 + \sqrt{5}}{2}\right)$ and $F\left(\frac{3 - \sqrt{5}}{2}\right)$.

11. If $f(x) = \frac{x + 1}{x - 1}$, find $f(\sqrt{2})$ in simplest form.

12. If $f(y) = \frac{y + 2}{y - 1}$ and $g(y) = \frac{y - 2}{y + 1}$, find $\frac{f(y) + g(y)}{2 + f(y) \cdot g(y)}$ and express the result in simplest form.

13. If $F(x, y) = 2x^2 + 3xy - 2y^2$, find $F(1, 2), F(-1, -2), F(2, 3), F(-2, -3)$.

14. If $F(x, y) = x^3 + x^2y + xy^2 + y^3$, show that $F(y, x) = F(x, y)$ and that $F(-x, -y) = -F(x, y)$.

15. If $G(x, y) = \frac{x + y}{x - y}$, find $G(\sqrt{3}, \sqrt{2})$ in simplest form.

16. If $f(x) = x^2 + 5x - 2$, find $\frac{f(x + h) - f(x)}{h}$. This operation is performed in the calculus in the course of obtaining *the derivative of* $f(x)$.

17. If $f(x) = \frac{1}{x + 1}$, find $\frac{f(x + h) - f(x)}{h}$.

18. If $F(x, y) = x^3 - 5x^2y + 3xy^2 - 3y^3$, show that $F(kx, ky) = k^3F(x, y)$.

19. Generalize Ex. 18 by showing that if $F(x, y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_{n-1}xy^{n-1} + a_ny^n$, where the a's are constants, then

$F(kx, ky) = k^nF(x, y)$. This is a test for the *homogeneity of a function* and shows that $F(x, y)$ is a homogeneous polynomial of degree n (Sec. 2.2).

20. If $F(x, y) = 4x^2 + 9y^2$, show that $F(x, y) = F(-x, y) = F(x, -y) = F(-x, -y)$. These results are illustrations of tests used in analytic geometry to determine various types of *symmetry of curves*.

21. If $y = f(x) = \dfrac{2x - 1}{3x - 2}$, show that $x = f(y)$.

22. If $x = g(y) = \dfrac{5y + 4}{y - 5}$, show that $y = g(x)$.

23. If $f(x) = \dfrac{3x + 2}{2x - 3}$, show that $f[f(x)] = x$.

24. If $g(y) = \dfrac{4y - 2}{3y - 4}$, show that $g[g(y)] = y$.

25. If $y = f(x) = \dfrac{2x + 1}{2x - 1}$, find $f(y)$ in terms of x.

3.6. CLASSIFICATION OF FUNCTIONS

In Sec. 3.4 we discussed various *types* of functions. We will now consider the division of functions into various *classes* according to their *forms*. We first lay down the following

Definition. A function of the variable x is said to be *algebraic* if it involves x in only a finite number of one or more of the six operations of algebra.

Thus, examples of algebraic functions of x are

$$x^2 - 2x + 5, \quad \frac{4x^2 - 3x + 2}{x^3 + 2x^2 - 9}, \quad \text{and} \quad \sqrt[3]{3x + 5}.$$

NOTE 1. The student should compare this definition with the fundamental definition given in Sec. 1.6.

This definition of algebraic function is sufficient for all the purposes of this book and for almost all other situations the student may encounter in his later work. However, it should be pointed out that this definition does not include *all* algebraic functions, as explained in Note 2 on page 72.

A *rational integral function of x* is a function of the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n,$$

where n is a positive integer or zero and $a_0, a_1, \cdots, a_n$ are any constants. We ordinarily speak of such a function as a *polynomial* in x. In particular, if $a_0 \neq 0$, the function or polynomial is said to be of *degree n*. (See Sec. 2.2.)

A *rational function of* x is the quotient of one rational integral function of x by another which is different from zero. Thus, if $f(x)$ and $g(x)$ are both rational integral functions of x, if $g(x) \neq 0$, and if

$$(1) \qquad R(x) = \frac{f(x)}{g(x)},$$

then $R(x)$ is a rational function of x.

An algebraic expression which cannot be put in the form (1) above is said to be an *irrational function*. Thus, $\sqrt{x+1}$ and $\dfrac{x - \sqrt{2 + x^3}}{x+1}$ are irrational functions.

The three preceding definitions may be readily extended to functions of several variables. Thus, $2x^2 + 3xy - 4y^2$ is a rational integral function of x and y, $\dfrac{2x^2 + 3xy - 4y^2}{x^3 + 3x^2y - 2y^3}$ is a rational function of x and y, and $\sqrt{x+y}$ is an irrational function of x and y.

Now consider that x and y are connected by the implicit relation

$$(2) \quad y^m + R_1(x)y^{m-1} + R_2(x)y^{m-2} + \cdots + R_{m-1}(x)y + R_m(x) = 0,$$

where m is a positive integer and $R_1(x)$, $R_2(x)$, $\cdots$, $R_m(x)$ are rational functions of x. If the relation between the two variables x and y is of the form (2), or may be made to assume such a form, then y is said to be an *algebraic function* of x. Thus each of the relations $x^2 + y^2 = 1$, $y^2 = \dfrac{x^3}{x-2}$, and $x^{1/2} + y^{1/2} = 1$ expresses y as an algebraic function of x.

NOTE 2. Advanced treatises show that if $m \geq 5$ in relation (2), it is impossible except for special conditions to express y explicitly in terms of x by a finite number of one or more of the six operations of algebra. Nevertheless, even in such cases y is said to be an algebraic function of x. This is the reason for stating in Note 1 that our first definition did not include *all* algebraic functions as given by the second definition. However, as previously indicated, the first definition will be sufficient for our purposes.

All functions which are not algebraic are called *transcendental functions*. Examples of such functions are the trigonometric, logarithmic, and exponential functions.

3.7. THE LINEAR COORDINATE SYSTEM

We now give additional significance to the properties of real numbers by introducing the idea of the correspondence between a geometric point and a real number. Consider, as in Fig. 1, a straight line $X'X$ whose positive direction is from left to right as indicated by the arrow, and let O be a fixed point on this line. Next we adopt a convenient length as a unit of measure; thus, if A is a point on $X'X$ distinct from O and to the right of it, then the length OA may be considered the unit of length. If P is any point on $X'X$ and to the *right* of O such that the length OP contains our adopted unit of length x times, we shall say that the point P *corresponds* to the *positive* number x. Similarly, if P' is any point on $X'X$ and to the *left* of O such that OP' has a length of x' units, we shall say that the point P' *corresponds*

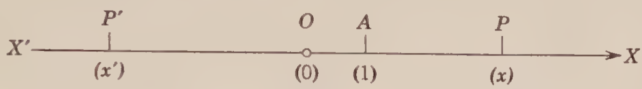


Figure 1

to the *negative* number x'. In this way any given real number x may be represented by a point P on the line $X'X$. And, conversely, any given point P on the line $X'X$ represents a real number x whose numerical value is equal to the length of OP and whose sign is positive or negative according to whether P is to the right or left of O.

Accordingly, we have constructed a scheme whereby a reciprocal correspondence is established between geometric points and real numbers. Such a scheme is called a *coordinate system*, the basic concept of *analytic geometry*, first introduced in 1637 by the French mathematician René Descartes (1596–1650). In the particular case under consideration, since the points all lie on the same straight line, the system is called a *one-dimensional* or *linear coordinate system*. Referring to Fig. 1, the line $X'X$ is called the *axis* and the point O the *origin* of the linear coordinate system. The real number x corresponding to the point P is called the *coordinate* of the point P and is represented by (x). Obviously, in accordance with the conventions adopted, the origin O has the coordinate (o) and the point A has the coordinate (1). The point P with the coordinate (x) is said to be the *geometric* or *graphic representation* of the real number x, and the coordinate (x) is said to be the *analytic representation* of the point P.

We note, furthermore, that the reciprocal correspondence of the linear coordinate system is *unique*, for to each real number there corresponds one and only one point on the axis, and to each point on the axis there corresponds one and only one real number.

The student will note that *real* numbers only are considered in this coordinate system. The geometric representation of the complex number will be discussed later in our study of that number (Chap. 8).

We are now in a position to give a geometric interpretation of the meaning of the algebraic statement that one number is greater than another (Sec. 2.4). Thus, let the real numbers a and b be the respective coordinates of the points P and Q. If the point P lies to the *right* of the point Q on the coordinate axis, then $a > b$. The student should illustrate this statement by using various pairs of real numbers, both positive and negative.

Finally, we note that the linear coordinate system is a convenient means for representing the real numbers in the range of a variable (Sec. 3.2). But, in a functional relation (Sec. 3.3), if the linear coordinate system is used to represent the real values in the range of the independent variable x, some provision must be made for representing the corresponding values of the function or dependent variable y. This implies that for the geometric representation of the functional relation, another dimension is required.

3.8. THE RECTANGULAR COORDINATE SYSTEM

In a linear coordinate system, a point is restricted to lie on a single line, the axis. Therefore, we now consider a coordinate system in which a point is free to move to all positions in a plane. This is called a *two-dimensional* or *planar coordinate system*. There are various types of planar coordinate systems but the one that we shall use is the *rectangular coordinate system*, which is illustrated in Fig. 2, where two directed lines $X'X$ and $Y'Y$, called the *coordinate axes*, are drawn *perpendicular* to each other. The horizontal line $X'X$ is termed the X-axis, the vertical line $Y'Y$ the Y-*axis*, and O their point of intersection, the *origin*. The coordinate axes divide the plane into four regions called *quadrants*, numbered as shown in Fig. 2. As indicated by the arrows, the positive direction of the X-axis is to the right; the positive direction of the Y-axis is upward.

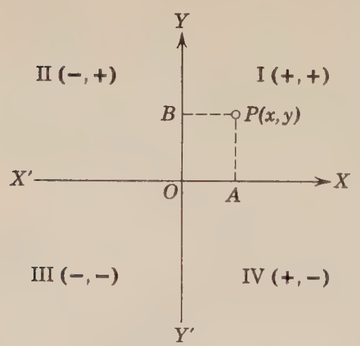


Figure 2

Any point P in the plane may be definitely located by means of the rectangular system. Thus, draw PA perpendicular to the X-axis and PB perpendicular to the Y-axis. The length of the line segment OA is denoted by x and called the *abscissa* of P; the length of the line segment OB is denoted by y and called the *ordinate* of P. The two real numbers x and y are called the *coordinates* of P and are represented by (x, y). Abscissas measured along the X-axis to the right of O are positive and to the left, negative; ordinates measured along the Y-axis above O are positive and below, negative. The signs of the coordinates in all four quadrants are indicated in Fig. 2.

It is evident that each point P in the coordinate plane has one and only one set of coordinates (x, y). Conversely, any set of coordinates (x, y) determines one and only one point in the coordinate plane.

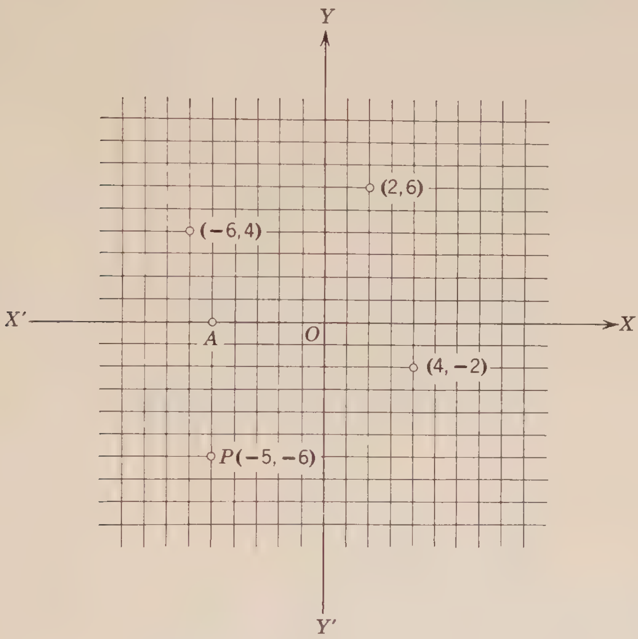


Figure 3

In general, $x \neq y$ for the coordinates (x, y) so that the point with the coordinates (x, y) is distinct from the point with the coordinates (y, x). Accordingly, it is important to write the coordinates in their proper order, the abscissa appearing in the first place and the ordinate in the second. For this reason a set of coordinates in the plane is called an *ordered* pair of real numbers. In view of our previous discussion, we may then say that *the rectangular coordinate system in the plane establishes a one-to-one correspondence between each point in the plane and an ordered pair of real numbers.*

The location of a point by means of its coordinates is termed *plotting* the point. For example, to plot the point $(-5, -6)$, we first obtain the point A on the X-axis which is 5 units to the left of O; then from A, on a line parallel to the Y-axis, we lay off 6 units below the X-axis, thus arriving at the point $P(-5, -6)$. This is shown in Fig. 3, where the points (2, 6), (−6, 4), and (4, −2) are also plotted.

The plotting of points is greatly facilitated by the use of rectangular coordinate paper which is ruled into equal squares by lines parallel to the coordinate axes. An illustration of such paper is given in Fig. 3. It is recommended that the student employ coordinate paper whenever very accurate plotting is required.

Once again, the student will note that this coordinate system makes no provision for the complex numbers of algebra. Hence, if either coordinate of a point is a complex number, such point has no existence in the rectangular coordinate system.

3.9. GRAPHICAL REPRESENTATION OF FUNCTIONS

We will now see how to use the rectangular coordinate system to give a geometric or pictorial representation of a functional relation. This process has a distinct advantage in that it presents to the eye a picture of the behavior of a given function of a single variable.

Let us consider the functional relation or equation

$$y = f(x), \tag{1}$$

which states that the dependent variable y is a function of the independent variable x. This means that for each value assigned to x, one or more corresponding values of y are determined. Each such pair of corresponding values of x and y is said to *satisfy* equation (1). We now take each of these pairs of *real* values as the *coordinates* (x, y) *of a point in the rectangular coordinate system.* This convention is the basis of

Definition 1. The totality of points, and *only* those points, whose coordinates satisfy an equation (1) is called the *locus or graph of the equation.*

Another convenient expression is given by

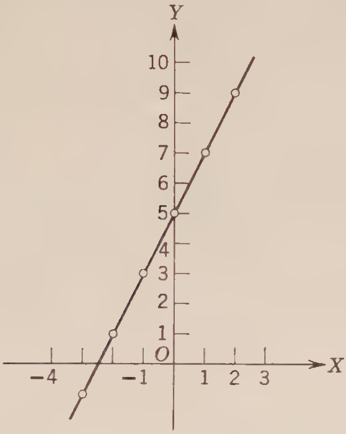
Definition 2. Any point whose coordinates satisfy an equation (1) is said to *lie on the locus of the equation.*

That is, if the coordinates of a point satisfy an equation that point lies on the locus of the equation, and conversely, if a point lies on the locus of an equation, its coordinates satisfy the equation. This is, of course, the statement of a necessary and sufficient condition (Sec. 2.15).

Since the coordinates of the points of a locus are restricted by its equation, such points will in general be located in positions which, taken together, form a definite path called a *curve* as well as a graph or locus.

Example 1. Plot the graph of the function $2x + 5$.

SOLUTION. Let $y = 2x + 5$. Since there are infinitely many pairs of corresponding values of x and y which satisfy this relation, we select only a sufficient number to give an adequate graph, as shown in Fig. 4. Each pair of corresponding values, taken as the coordinates of a point, is plotted as shown. A smooth curve is then drawn through these points and



x	y
0	5
1	7
2	9
−1	3
−2	1
−3	−1

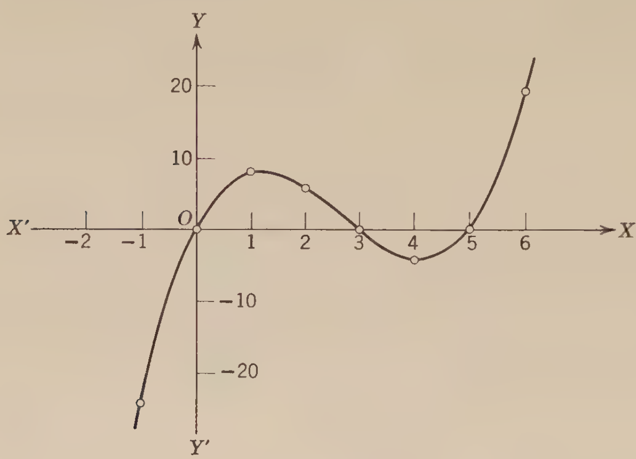
Figure 4

represents the graph of the given function. These points *appear* to lie on a straight line. As a matter of fact, it is definitely proved in analytic geometry that the locus of this function is a straight line.

Example 2. Plot the graph of the function $x^3 - 8x^2 + 15x$.

SOLUTION. Let $y = x^3 - 8x^2 + 15x$. By assigning certain values to x and computing the corresponding values of y, we obtain the coordinates of a suitable number of points, as shown in Fig. 5. Plotting these points and drawing a smooth curve through them we obtain the graph shown in Fig. 5. But in doing this we assumed that the graph between any two successive plotted points necessarily had the appearance resulting from the smooth curve drawn connecting the points. Although this is true for the particular graph under consideration here, it is not necessarily true for the graphs of all algebraic functions. However, for the polynomial function of a single variable, of which this function is an example, it is shown in the calculus that the graph is a smooth continuous curve.

It is appropriate at this time to call attention to a very important term, namely, the *zero of a function*. By a *zero of* $f(x)$, we mean a value of x such that the corresponding value of $f(x)$ is zero. Thus, -1 is a zero of the function $2x + 2$. Graphically, the *real* zeros of $f(x)$ are the abscissas of the points where the graph crosses the X-axis. Thus, as shown in Fig. 5, the real zeros of the function $x^3 - 8x^2 + 15x$ are 0, 3, and 5. We shall see subsequently that the determination of the zeros of functions is a basic problem of great importance in algebra.



x	y
0	0
1	8
2	6
3	0
4	−4
5	0
6	18
−1	−24

Figure 5

NOTE. The student will observe that graphical representation has been restricted here to algebraic functions of a *single* variable. For functions of several variables, the problem becomes more complicated. For example, for functions of two independent variables, a three-dimensional coordinate system is required. This is primarily a problem of solid analytic geometry and will not be discussed in this book.

EXERCISES. GROUP 10

In each of Exs. 1–18, plot the graph of the given function.

1. x.
2. $x + 1$.
3. $2x - 1$.
4. x^2.
5. x^3.
6. x^4.
7. $2x^2 - 1$.
8. $1 - x^2$.
9. $4 - x^2$.
10. $\sqrt{4 - x^2}$.
11. $-\sqrt{4 - x^2}$.
12. $\sqrt{x^2 - 1}$.
13. $\pm\sqrt{x^2 - 1}$.
14. $x^2 - 5x$.
15. $x^2 - 4x + 1$.
16. $1 - 4x - x^2$.
17. $x^3 - x$.
18. $x^3 + x$.

In each of Exs. 19–27, plot the graph of the given equation.

19. $x + y = 1$.
20. $x - y = 1$.
21. $x + 5 = 0$.
22. $y - 2 = 0$.
23. $y = x^2 + 1$.
24. $x^2 + y = 9$.
25. $x^2 + y^2 = 1$.
26. $y = x^3 - 4x$.
27. $x^3 + y = 8$.

In each of Exs. 28–33, plot the graph of the given function and find its real zeros.

28. $x^2 - 1$.
29. $x^2 - x - 2$.
30. $x^2 - 2x - 4$.
31. $x^2 + 2x - 2$.
32. $x^3 + 2x^2 - x - 2$.
33. $x^3 - 3x^2 + 2x$.

In each of Exs. 34 and 35, show by means of a graph that the given function has no real zeros.

34. $x^2 + 5$.
35. $x^2 - 2x + 3$.

4

The linear function

4.1. INTRODUCTION

At the close of Chapter 3 we stated that the determination of the zeros of functions is a basic problem of great importance in algebra. One such function is the rational integral function of x of degree n,

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n, \qquad a_0 \neq 0,$$

where n is a positive integer and $a_0, a_1, \cdots, a_n$ are any constants but $a_0 \neq 0$. In this chapter we consider the particular case where $n = 1$. The function then takes the form

$$(1) \qquad a_0x + a_1, \qquad a_0 \neq 0.$$

As previously noted in Example 1 of Sec. 3.9, it is proved in analytic geometry that the graph of the function (1) is a straight line. Accordingly, the function (1) is appropriately called the *linear function*.

4.2. THE EQUATION

An *equation* is a statement of equality between two expressions. These expressions are called the *members* or *sides* of the equation. Thus, in the equation

$$x^2 + 4 = 5x,$$

the expression $x^2 + 4$ is called the *first* or *left member* (side) and $5x$ is called the *second* or *right member* (side).

We first consider two types of equations, the *identical equation* and the *conditional equation*.

An *identical equation*, or simply an *identity*, is an equation in which both members are equal for *all* values of the variables for which these members are defined. In an identity the equality sign = is often replaced by the identity symbol ≡, read "is identically equal to." Examples of identities are

(1) $$(a - b)^2 \equiv a^2 - 2ab + b^2,$$

(2) $$\frac{x}{x - 1} \equiv 1 + \frac{1}{x - 1}.$$

Identity (1) is true for *all* values of a and b; identity (2) is true for all values of x except 1.

A *conditional equation*, or simply an *equation*, is an equation in which both members are equal only for certain particular values of the variables. Examples of conditional equations are

(3) $$x^2 - 5x + 4 = 0,$$

(4) $$x + y = 5.$$

Equation (3) is true only for $x = 1$ and $x = 4$ and for no other values of x, that is, only on *condition* that $x = 1$ or 4. Equation (4) is true for infinitely many sets of values of x and y but not for *all* sets of values; thus, (4) is true for $x = 1$, $y = 4$ and for $x = 2$, $y = 3$, and so on, but not for other sets such as $x = 3$, $y = 3$ and $x = 4$, $y = 2$, and so on.

NOTE. In an equation there are symbols whose values are either *known* or assumed known, while other symbols represent *unknown values*. Thus, in (3), x is an unknown number or variable, the numbers 4 and 5 being, of course, known; in (4), both x and y are unknown numbers, 5 being known.

If an equation is reduced to an identity for certain particular values assigned to the variables, the equation is said to be *satisfied* for those values. (See Sec. 3.9.) Thus, equation (3) is satisfied when x is assigned the value 1, since the equation then reduces to the identity $1 - 5 + 4 = 0$. Also, equation (4) is satisfied for $x = 1$, $y = 4$, since it then reduces to the identity $1 + 4 = 5$.

Any number which satisfies an equation in one unknown or variable is called a *root* or *solution* of that equation. Thus, 1 is a root of equation (3), which may be written in the form

$$f(x) = x^2 - 5x + 4 = 0.$$

Hence, 1 is a zero of $f(x)$ (Sec. 3.9). In general, *a zero of the function* $f(x)$ *is a root or solution of the equation* $f(x) = 0$.

A set of values of the unknowns which satisfies an equation containing two or more unknowns or variables is called a *solution* of that equation.

Thus, $x = 1$, $y = 4$ is a solution of (4). Evidently equation (4) has many solutions (Sec. 3.9).

4.3. EQUIVALENT EQUATIONS

The discussion in this section is restricted to a single equation in one unknown or variable x, conveniently represented by

(1) $$f(x) = 0.$$

We shall discuss the determination of the roots of (1); this is called the *solution* of equation (1). The method generally consists in transforming (1) into another equation, say,

(2) $$F(x) = 0,$$

whose roots are more easily obtained than the roots of (1). Obviously this procedure is applicable if, and only if, the roots of equation (2) are the same as the roots of equation (1); these two equations are then said to be *equivalent*.

Thus, $x - 2 = 0$ and $2x = 4$ are equivalent, each having the single root 2. But $x - 2 = 0$ and $x^2 - 4 = 0$ are not equivalent because the st equation has the single root 2 and the second equation has the two roots ± 2.

We next consider those operations on a given equation which lead to an equivalent equation. We recall that in Chapter 2 we stated the equality law for each of the four operations of addition, subtraction, multiplication, and division. From these laws we may show that a given equation may be transformed into an equivalent equation by any of the following operations:

1. If the same *expression* is added to or subtracted from both members of a given equation, the resulting equation is equivalent to the given equation.
2. If both members of a given equation are multiplied or divided by the same *nonzero known number*, the resulting equation is equivalent to the given equation.

Thus, with respect to operation 1 above, consider that the expression $g(x)$ is added to both sides of (1) so that by the equality law for addition, we have the equation

(3) $$f(x) + g(x) = g(x).$$

Let r be any root of (1) so that $f(r) \equiv 0$. Substituting r for x in (3), we obtain the identity

$$0 + g(r) \equiv g(r),$$

so that r is a root of (3).

Conversely, let s be any root of (3) so that we have the identity

$$f(s) + g(s) \equiv g(s),$$

whence, by the equality law for subtraction, we have the identity

$$f(s) \equiv 0,$$

so that s is a root of (1). Hence equations (1) and (3) are equivalent.

Similarly, we may establish the validity of the other operations leading to equivalent equations. However, it must be noted that there is a distinction among these operations, namely, that in addition and subtraction we may add or subtract any *expression*, which may include *both* variables and constants, but in multiplication and division we may multiply and divide only by a nonzero *constant*.

If both members of the given equation are multiplied by an expression containing the variable, the new equation may have one or more roots which are not roots of the given equation. These new roots are called *extraneous roots* and the new equation is called *redundant* with respect to the given equation.

As an illustration, consider the equation

$$x = 3 \tag{4}$$

with the root 3. If we multiply both members of (4) by $x - 2$, we obtain the equation

$$x(x - 2) = 3(x - 2) \tag{5}$$

with the roots 2 and 3. Hence equations (4) and (5) are not equivalent, 2 is an extraneous root, and equation (5) is redundant with respect to equation (4).

Another case may be noted. If both members of (4) are squared, we obtain the equation $x^2 = 9$ with the roots ± 3. Hence this operation has introduced the extraneous root -3.

If both members of the given equation are divided by an expression containing the variable, the new equation may lack one or more of the roots of the given equation. The new equation is then said to be *defective* with respect to the given equation.

As an illustration of this situation, divide both members of equation (5) by $x - 2$. We thus obtain the equation

$$x = 3. \tag{6}$$

Equation (5) has the roots 2 and 3 but equation (6) is defective with respect to equation (5), for it has only the root 3.

Consequently, care must be exercised when performing operations on a given equation so that we do not introduce any extraneous roots and also do not lose any valid roots. In all cases the student should make it a fixed rule to *check each root by substitution in the original equation*.

Finally, we call attention to an operation which is very simple but used frequently in the solution of equations. Consider any equation, say,

$$a + b = c - d, \tag{7}$$

where a, b, c, and d are individual terms. By the equality law for addition, if we add d to both members, we obtain the equation

$$a + b + d = c. \tag{8}$$

Comparing (7) and (8), we see that the term d has been transposed from the right to the left member by changing its sign.

Also, by the equality law for subtraction, if we subtract b from both members of (7) we obtain the equation

$$a = c - d - b. \tag{9}$$

Comparing (7) and (9), we see that the term b has been transposed from the left to the right member by changing its sign. Accordingly, we have the

Rule of transposition. Any term may be transposed from one member of an equation to the other by merely changing its sign.

4.4. THE LINEAR EQUATION IN ONE VARIABLE

If the linear function in one variable (Sec. 4.1) is set equal to zero, we have *the linear equation in one variable*,

$$ax + b = 0, \qquad a \neq 0, \tag{1}$$

where a and b are arbitrary constants.

As a first step in the solution of this equation, we transpose b to the right member, thus obtaining the equivalent equation

$$ax = -b.$$

Next, by dividing both sides by a, we obtain the equivalent equation and solution

$$x = -\frac{b}{a}.$$

If this value of x is substituted in (1) we obtain the identity

$$a\left(-\frac{b}{a}\right) + b = -b + b = 0.$$

We record this result as

Theorem 1. *The linear equation in one variable,*

$$ax + b = 0, \qquad a \neq 0,$$

has the unique solution
$$x = -\frac{b}{a}.$$

Accordingly, to solve a linear equation in one variable, we transpose, if necessary, all unknown terms to one side and all known terms to the other side of the equation.

Example 1. Solve the equation $ax + b^2 = a^2 + bx$, $a \neq b$.

SOLUTION. Here, of course, it is understood that the unknown quantity or variable is x, all other letters, therefore, being considered known constants. Accordingly, we proceed as follows:

By transposition, $\qquad ax - bx = a^2 - b^2.$

Combining terms, $\qquad (a - b)x = a^2 - b^2.$

Dividing by $a - b, a \neq b$, $\qquad x = a + b.$

We *check* our solution by substituting the root $a + b$ for x in the *original* equation. We obtain

$$a(a + b) + b^2 = a^2 + b(a + b),$$

or the identity, $\qquad a^2 + ab + b^2 = a^2 + ab + b^2.$

Example 2. Solve the equation $\dfrac{5}{x + 2} - \dfrac{10}{x^2 - 4} = \dfrac{1}{2 - x}$.

SOLUTION. A *fractional equation*, such as this, is first *cleared of fractions* by multiplying both sides by the L.C.D. of the fractions (Sec. 2.11). If the L.C.D. is a number, the resulting equation is equivalent to the original equation, but if the L.C.D. contains the variable, we may introduce extraneous roots (Sec. 4.3) which are the zeros of the L.C.D. In this problem, such possible extraneous roots are ± 2.

Multiplying both sides of the given equation by the L.C.D., $x^2 - 5$, we obtain

$$5(x - 2) - 10 = 1(-x - 2),$$

$$5x - 10 - 10 = -x - 2,$$

whence $\qquad 6x = 18,$

and $\qquad x = 3.$

Now 3 is not an extraneous root but, for *accuracy*, it should be checked in the *original equation*. This is left as an exercise to the student.

EXERCISES. GROUP 11

In each of Exs. 1–20, solve the given equation and check the solution.

1. $3x - 2 = 3 - 2x.$
2. $4 - 2x = 3x + 14.$
3. $\dfrac{x}{2} - \dfrac{3x}{5} = \dfrac{x - 6}{2}.$
4. $\dfrac{x + 3}{2} - \dfrac{2 - 3x}{7} = \dfrac{4x}{3}.$
5. $3x - (x + 3) = x + 4.$
6. $x - [4 - (x + 1)] = 4x - 15.$
7. $2(2x - a) - (a - 2x) = 3x.$
8. $ax - c + bx - d = 0.$
9. $(m + n)x + (m - n)x = 2m^2.$
10. $ax - b^3 = a^3 - bx.$
11. $\dfrac{x}{a} + \dfrac{x}{b} = \dfrac{a + b}{a}.$
12. $\dfrac{x}{a} - \dfrac{x}{a + b} = \dfrac{1}{a + b}.$
13. $(x + a^2)(x + b^2) = (x + ab)^2.$
14. $(x + 1)(x - 2) = x^2 + 6.$
15. $\dfrac{2x}{x - 1} - 2 = \dfrac{3}{x + 1}.$
16. $\dfrac{1}{x + 3} + \dfrac{1}{x} = \dfrac{2}{x + 1}.$
17. $\dfrac{1}{x + 2} + \dfrac{1}{x - 3} = \dfrac{3}{x^2 - x - 6}.$
18. $\dfrac{x + b}{x - b} = \dfrac{x}{x + b}.$
19. $\dfrac{1}{x} - \dfrac{1}{r} = \dfrac{1}{s} - \dfrac{1}{x}.$
20. $\dfrac{a^2 + b^2}{2bx} - \dfrac{b}{x} = \dfrac{a - b}{2bx^2}.$

In each of Exs. 21–24, solve the given equation for y in terms of x and for x in terms of y.

21. $3x + 2y = 6.$
22. $4x - 5y = 10.$
23. $bx + ay = ab.$
24. $ax + by + c = 0.$

In each of Exs. 25–28, solve the given relation for the indicated letter in terms of the remaining letters.

25. $A = P(1 + rt);\ t.$
26. $a_n = a_1 + (n - 1)d;\ d.$
27. $s = s_0 + v_0 t + \frac{1}{2}gt^2;\ v_0.$
28. $\dfrac{1}{f} = \dfrac{1}{p} + \dfrac{1}{q};\ q.$

29. Show that each of the equations $\sqrt{x + 1} = \sqrt{x}$ and $\sqrt{x - 1} = \sqrt{x + 1}$ has no solution.

30. Show that the following equation has no solution:

$$\frac{x + 1}{x - 1} + 4 = \frac{4x^2}{x^2 - 1} + \frac{x - 1}{x + 1}.$$

31. Solve and check: $\dfrac{x - 6}{x - 7} - \dfrac{x - 5}{x - 6} = \dfrac{x - 2}{x - 3} - \dfrac{x - 1}{x - 2}.$

32. Prove that if the same expression is subtracted from both members of a given equation, the resulting equation is equivalent to the given equation.

33. Prove that if both members of a given equation are multiplied by the same nonzero number, the resulting equation is equivalent to the given equation.

34. Prove that if both members of a given equation are divided by the same nonzero number, the resulting equation is equivalent to the given equation.

35. Show that it is impossible for the linear equation $ax + b = 0$, $a \neq 0$, to have two distinct solutions.

4.5. PROBLEMS SOLVABLE BY A LINEAR EQUATION

It is possible to solve a great variety of problems by means of the linear equation in one variable. The procedure generally consists in assigning some letter, say x, to represent the unknown quantity (or one of the unknown quantities). The next step is to construct an equation involving x and meeting the conditions of the problem. The final step is then the solution of this equation for the required value of x. In this whole procedure it is important for the student to understand that *the letter x always represents a number*. It is also important to check the solution by showing that it satisfies the conditions of the problem.

Example 1. A certain piece of work can be completed by A alone in 4 days and by B alone in 6 days. How long will it take them to complete the work together?

SOLUTION. Let x = *number* of days required.

Then $\frac{1}{x}$ = part of work both can do in 1 day,

$\frac{1}{4}$ = part of work A can do in 1 day,

$\frac{1}{6}$ = part of work B can do in 1 day.

Hence, $$\frac{1}{x} = \frac{1}{4} + \frac{1}{6}.$$

Multiplying by $12x$, $\quad 12 = 3x + 2x,$

whence $$x = \frac{12}{5} = 2\frac{2}{5} \text{ days required.}$$

CHECK. In $2\frac{2}{5}$ days, the part of the work done by A is $\frac{12}{5} \cdot \frac{1}{4} = \frac{3}{5}$, and the part done by B is $\frac{12}{5} \cdot \frac{1}{6} = \frac{2}{5}$; the sum of these parts is $\frac{3}{5} + \frac{2}{5} = 1$, the entire work.

Example 2. A mixture of 16 qt of alcohol and water is 25 per cent alcohol. How many quarts of alcohol must be added to obtain a mixture which is 50 per cent alcohol?

SOLUTION. Let x = *number* of quarts of alcohol to be added. Then $16 + x$ = number of quarts in final mixture. In the original mixture there are $\frac{1}{4} \cdot 16 = 4$ qt of alcohol. Then $4 + x$ = number of quarts of alcohol in final mixture.

Hence, $$\frac{4 + x}{16 + x} = \frac{1}{2},$$

whence $$8 + 2x = 16 + x,$$

and $x = 8$ = number of quarts of alcohol added.

CHECK. The final mixture = 16 + 8 = 24 qt.
Total alcohol in final mixture = 4 + 8 = 12 qt = 50 per cent of 24 qt.

EXERCISES. GROUP 12

In each of the following problems, the result(s) should be checked.

1. A wire 21ft long is divided into two parts such that the length of one part is three fourths of the length of the other part. Find the length of each part.

2. The denominator of a fraction exceeds the numerator by 2. If each term of the fraction is increased by 5, the value of the fraction becomes $\frac{4}{5}$. Find the fraction.

3. Find three consecutive numbers whose sum is equal to 21.

4. Find three consecutive even numbers whose sum is equal to 36.

5. Find two numbers whose sum is 24 and whose difference is 6.

6. Eight years ago a man was 7 times as old as his son, but now he is only 3 times as old. Find the present age of each.

7. If $\frac{1}{5}$ of A's age is increased by $\frac{1}{4}$ of what it was 10 years ago, the sum is equal to $\frac{1}{3}$ of his age 10 years hence. Find A's age.

8. Divide 40 into two parts such that if the quotient of the larger by the smaller is diminished by the quotient of the smaller by the larger, the difference is equal to the quotient of 16 by the smaller.

9. Divide 72 into three parts such that $\frac{1}{2}$ of the first part, $\frac{1}{3}$ of the second part, and $\frac{1}{4}$ of the third part are all equal.

10. The units' digit of a two-digit number exceeds the tens' digit by 5. If the digits are reversed, the new number divided by the original number is equal to $\frac{8}{3}$. Find the original number.

11. If the side of a square is decreased by 1 ft, its area is decreased by 39 sq ft. Find the length of the side of the original square.

12. The length (ft) of a room is equal to 3 times its width. If the length is decreased by 5 ft and the width is increased by 2 ft, the area of the room remains unchanged. Find the dimensions of the room.

13. A certain piece of work can be completed by A alone in 3 hours, by B alone in 4 hours, and by C alone in 6 hours. How long will it take them all together to complete the work?

14. One pipe can fill a tank in 2 hours, a second pipe can fill it in 3 hours, and a third pipe can empty it in 6 hours. If all three pipes are open, how long will it take to fill the tank, which is initially empty?

15. A and B can do a certain piece of work in 8 hours and A alone can do it in 12 hours. How long will it take B alone to do it?

16. A alone can paint a house in 8 days and B alone can do it in 6 days. How long will it take B alone to finish the job after both A and B have worked 3 days?

17. A can do a piece of work in 4 hours and B can do it in 12 hours. B starts the work but after a time is replaced by A so that the entire work is finished exactly 6 hours after the start. How long did B work?

18. A crew can row at the rate of 9 mi per hour in still water. If it takes them twice as long to row a certain distance against the current as it does to row the same distance with the current, find the rate of the current.

19. A and B start at the same time from two towns to walk toward each other. If B walks 1 mi per hour faster than A, they meet in 6 hours. If A walks as fast as B, they meet in $5\frac{1}{4}$ hours. Find the distance between the two towns.

20. A motor boat can go 10 mi downstream in the same time that it goes 6 mi upstream. If its rate each way is decreased 4 mi per hour, its rate downstream will be twice its rate upstream. Find its rate downstream.

21. A can walk a certain distance in 20 minutes and B can walk the same distance in 30 minutes. If A starts 5 minutes after B, how long will B have been walking when A overtakes him?

22. How many quarts of alcohol 20 per cent pure and of alcohol 30 per cent pure must be mixed together to obtain 100 qt of alcohol 25 per cent pure?

23. How many ounces of silver 60 per cent pure and of silver 90 per cent pure must be mixed together to obtain 6 oz of silver 80 per cent pure?

24. How many quarts of cream containing 25 per cent of butter fat must be added to 80 qt of milk containing 3 per cent of butter fat to obtain a mixture containing 5 per cent of butter fat.

25. A tank contains 100 lb of brine having a salt content of 5 per cent. How many pounds of pure water must be boiled off in order to obtain brine with a salt content of 8 per cent?

26. At what time between 3 and 4 o'clock are the hands of a clock together?

27. At what time between 3 and 4 o'clock are the hands of a clock opposite each other?

28. At what time between 4 and 5 o'clock are the hands of a clock at right angles to each other?

29. A and B can pave a walk in 2 days; B and C can do it in $1\frac{1}{5}$ days; and A and C in $1\frac{1}{2}$ days. In what time can each alone do the work?

30. A boy has a certain amount of money. If he buys 10 pencils, he will have 10 cents left; if he buys 4 notebooks, he will have 20 cents left; and if he buys 4 pencils and 3 notebooks, he will have 10 cents left. How much money does he have?

4.6. THE LINEAR EQUATION IN TWO VARIABLES

The linear function in two variables is represented by the expression

$$ax + by + c, \qquad ab \neq 0,$$

where a, b, and c are arbitrary constants and the restriction $ab \neq 0$ means that neither a nor b can be equal to zero (Theorem 19, Sec. 2.15).

If this function is placed equal to zero, we have *the linear equation in two variables*,

$$(1) \qquad ax + by + c = 0, \qquad ab \neq 0.$$

Solving (1) for y in terms of x and also for x in terms of y, we obtain, respectively, the equivalent equations

$$(2) \qquad y = -\frac{a}{b}x - \frac{c}{b}, \qquad b \neq 0,$$

$$(3) \qquad x = -\frac{b}{a}y - \frac{c}{a}, \qquad a \neq 0.$$

As noted previously (Sec. 4.2), any one of these three equations has infinitely many solutions. Such equations are therefore said to be *indeterminate*. But in connection with the solution of a definite problem, we must have a unique result which, evidently, cannot be obtained from a *single* equation in *two* variables. Suppose, however, that in addition to equation (1), we have another linear relation in x and y. We can then solve this relation for y in terms of x and equate it to the value of y given by (2). We may thus obtain a single equation in x alone which has a unique solution (Theorem 1, Sec. 4.4). Similarly, by solving this relation for x in terms of y and using (3), we may obtain a single equation in y with a unique solution.

It appears, therefore, that for a definite or unique solution involving two or more variables, two or more linear equations are required. Such a group of equations is called a *system of linear equations*.

4.7. SYSTEM OF LINEAR EQUATIONS

Consider the system of two linear equations in two variables,

$$(1) \qquad a_1x + b_1y + c_1 = 0, \qquad a_1b_1 \neq 0,$$

$$(2) \qquad a_2x + b_2y + c_2 = 0, \qquad a_2b_2 \neq 0,$$

where x and y represent the same numbers simultaneously in *both* equations. For this reason the equations are often called *simultaneous*. A set of values of x and y which satisfy *both* equations is called a *common solution* of the system. A system having only *one* common solution is said to have a *unique* solution.

If this system has a unique solution, it may be obtained by *eliminating* one of the variables and then solving for the other variable. There are several ways of effecting this elimination. One method is by *substitution*, as indicated in the preceding article. Another is by *addition or subtraction* and will now be discussed.

If we multiply equations (1) and (2), respectively, by the arbitrary constants or parameters k_1 and k_2, we obtain the *equivalent* equations

$$k_1(a_1x + b_1y + c_1) = 0,$$
$$k_2(a_2x + b_2y + c_2) = 0.$$

Adding these equations, we obtain

$$k_1(a_1x + b_1y + c_1) + k_2(a_2x + b_2y + c_2) = 0,$$

or

$$(3) \qquad (k_1a_1 + k_2a_2)x + (k_1b_1 + k_2b_2)y + (k_1c_1 + k_2c_2) = 0,$$

where k_1 and k_2 may assume any values except that both may not be zero simultaneously. Equation (3) is then called *a linear combination* of equations (1) and (2).

Suppose that the system (1) — (2) has a unique solution, say $x = x_1$, $y = y_1$. Then, from equations (1) and (2), we have the relations

$$(4) \qquad a_1x_1 + b_1y_1 + c_1 = 0,$$
$$(5) \qquad a_2x_1 + b_2y_1 + c_2 = 0.$$

If we now let $x = x_1$ and $y = y_1$ in (3), we find in view of (4) and (5) that it reduces to

$$k_1 \cdot 0 + k_2 \cdot 0 = 0,$$

which holds for all values of k_1 and k_2. Hence, *a unique solution of* (1) *and* (2) *is also a solution of* (3).

To effect the solution from (3), therefore, we need merely select those values of k_1 and k_2 which will eliminate one of the variables. Thus, to eliminate y from (3), we select values of k_1 and k_2 such that $k_1b_1 = -k_2b_2$.

Example 1. Solve and check the system

$$3x - 2y = 1, \qquad 2x + 3y = 18.$$

Illustrate the result graphically.

SOLUTION. If we multiply the first equation by 3 and the second equation by 2, we obtain the respective equivalent equations,

$$9x - 6y = 3,$$
$$4x + 6y = 36.$$

Adding, $13x = 39$, whence $x = 3$.

Similarly, we may obtain y by a suitable linear combination. However, it is probably simpler to substitute $x = 3$ in the first equation and solve for y. Thus,

$$9 - 2y = 1, \text{ whence } y = 4.$$

Hence, the solution is $x = 3$, $y = 4$. It may be checked by substitution in *each* of the given equations. Thus,

$$3(3) - 2(4) = 9 - 8 = 1,$$
$$2(3) + 3(4) = 6 + 12 = 18.$$

In Sec. 3.9 we noted that the graph of the linear equation in two variables is a straight line. The graphs of the two given equations are shown in Fig. 6. Their point of intersection has the coordinates (3, 4), representing the common solution of the two given equations. The graphs indicate that this solution is unique.

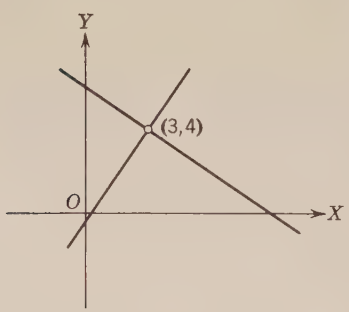


Figure 6

Up to this point we have considered only systems having a unique solution. Yet a system does not necessarily have a unique solution. For example, the system

$$x + y = 3, \qquad x + y = 2$$

has no common solution at all. On the other hand, the system

$$x + y = 2, \qquad 2x + 2y = 4$$

has infinitely many common solutions. In order to develop suitable criteria for determining the nature of the solutions of a linear system, we now consider the system

(6) $$a_1x + b_1y = c_1, \qquad a_1b_1 \neq 0,$$
(7) $$a_2x + b_2y = c_2, \qquad a_2b_2 \neq 0.$$

To eliminate y, we multiply (6) by b_2 and (7) by b_1, and then subtract, obtaining

$$a_1b_2x - a_2b_1x = b_2c_1 - b_1c_2,$$

whence
$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}.$$

Similarly, by eliminating x, we obtain

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

This solution is valid, of course, if and only if $a_1b_2 - a_2b_1 \neq 0$. If we substitute these values of x and y in the left member of (6), we obtain

$$a_1 \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1} + b_1 \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} = \frac{a_1b_2c_1 - a_1b_1c_2 + a_1b_1c_2 - a_2b_1c_1}{a_1b_2 - a_2b_1}$$
$$= \frac{c_1(a_1b_2 - a_2b_1)}{a_1b_2 - a_2b_1} = c_1,$$

that is, the solution satisfies (6). Similarly, the solution may be shown to satisfy (7). Hence the solution is unique and the system is said to be *consistent.*

We now investigate the situation when

$$a_1b_2 - a_2b_1 = 0,$$

whence $$a_1b_2 = a_2b_1 \quad \text{and} \quad \frac{a_1}{a_2} = \frac{b_1}{b_2}.$$

Let $a_1/a_2 = b_1/b_2 = r$, where $r \neq 0$ is a constant. Then $a_1 = ra_2$ and $b_1 = rb_2$. Substituting these values in (6), we have the equivalent equation

(8) $$ra_2x + rb_2y = c_1.$$

Multiplying both sides of (7) by r, we have the equivalent equation

(9) $$ra_2x + rb_2y = rc_2.$$

We note that the left members of (8) and (9) are identical. Hence, if $c_1 \neq rc_2$ we have a contradiction. In this case we cannot have a common solution, and the system is said to be *inconsistent* or *incompatible.*

But if $c_1 = rc_2$, equations (8) and (9) are identical and therefore equivalent to a single equation in two variables. In this case there are infinitely many solutions and the system is said to be *dependent.* If two equations are *not* reducible to the same form, they are said to be *independent.*

We record the preceding results in

Theorem 2. *The linear system*

$$a_1x + b_1y = c_1, \qquad a_1b_1 \neq 0,$$
$$a_2x + b_2y = c_2, \qquad a_2b_2 \neq 0,$$

has the unique solution

$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}, \qquad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1},$$

if and only if $a_1b_2 - a_2b_1 \neq 0$. *The system is then said to be consistent.*

If $a_1b_2 - a_2b_1 = 0$, *the system either has no solution and is said to be inconsistent or else has infinitely many solutions and is said to be dependent.*

NOTES 1. Results analogous to this theorem may be obtained for the general case of a linear system of n equations in n variables. Discussion of this, however, will be deferred until the study of determinants in a later chapter.

2. The student is advised not to use the results of this theorem as formulas for obtaining the solution of a linear system. It is preferable to employ the method of elimination used in Example 1 above.

Example 2. Examine the system

$$x - 2y = 4, \qquad 2x - 4y = -3$$

for a solution. Illustrate the result graphically.

SOLUTION. If we attempt to eliminate either variable, the other variable is also eliminated. When this occurs, the system must be examined more critically. Thus, if we multiply the first equation through by 2, we obtain the equivalent equation $2x - 4y = 8$, which, however, contradicts the second equation of the system. Hence the system is inconsistent and has no solution.

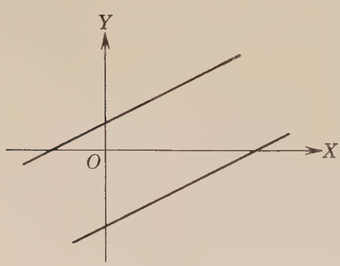


Figure 7

The graphs of the two given equations are shown in Fig. 7. Since they are parallel lines, they have no point in common; this is the geometric illustration of no solution for the given system.

Example 3. Examine the system

$$x - 2y = 4, \qquad 2x - 4y = 8$$

for a solution. Illustrate the result graphically.

SOLUTION. If we multiply the first equation by 2, we obtain the second equation. Hence the given system is dependent with infinitely many solutions. Both equations, being equivalent, are represented graphically by one line, the lower line in Fig. 7.

The method of elimination for obtaining the solution of a linear system may be readily extended to systems of three or more equations. This is illustrated in

Example 4. Solve and check the system

$$\begin{aligned} x + 2y - z &= 2, \\ 2x - y + z &= 3, \\ 2x + 2y - z &= 3. \end{aligned}$$

SOLUTION. We may reduce the given system to a system of two equations in two variables by eliminating one of the variables, say z. Thus, adding the first and second equations, we have

$$3x + y = 5,$$

and adding the second and third equations, we have

$$4x + y = 6.$$

The solution of this system of two equations is readily found to be $x = 1$, $y = 2$. Substituting these values of x and y in the first given equation, we have

$$1 + 4 - z = 2 \quad \text{or} \quad z = 3.$$

Hence the solution is $x = 1$, $y = 2$, $z = 3$. The student should check this solution by substitution in *each* of the given equations.

Some very important conclusions and observations are given in the following notes.

NOTES 3. We may infer from the preceding examples that the determination of a unique solution in n variables or unknowns requires a system of *n independent* equations.

4. We note that in a system of 2 independent equations, we may eliminate 1 variable, and in a system of 3 independent equations we may eliminate 2 variables. In general, *the elimination of n variables requires n + 1 independent equations.*

5. So far the number of equations in a given linear system has always been equal to the number of variables. If the number of equations *differs* from the number of variables, the system requires special treatment. Some particular cases of such systems are discussed in the later chapter on determinants, but the complete theory requires advanced treatment.

4.8. PROBLEMS SOLVABLE BY A SYSTEM OF LINEAR EQUATIONS

Many problems requiring the determination of two or more unknown quantities may often be solved by a system of linear equations. The unknown quantities are represented by letters, say x, y, etc. and a system of equations in these variables is then set up to meet the conditions of the problem. The solution of this system gives the values of the required quantities. The process is illustrated by several examples.

Example 1. The total cost of 5 textbooks and 4 fountain pens is \$32; the total cost of 6 textbooks and 3 fountain pens is \$33. Find the cost of each.

SOLUTION. Let $x =$ the cost of one textbook in dollars, and $y =$ the cost of one fountain pen in dollars.

Then from the conditions of the problem we have the two relations

$$5x + 4y = 32,$$
$$6x + 3y = 33.$$

The solution of this system is readily found to be $x = 4$, $y = 3$, that is, the cost of each textbook is \$4 and the cost of each fountain pen is \$3. These results may be easily checked. Thus, cost of 5 textbooks and 4 fountain pens $= 5(4) + 4(3) = \$32$. Cost of 6 textbooks and 3 fountain pens $= 6(4) + 3(3) = \$33$.

Example 2. Find two numbers such that the sum of their reciprocals is 5 and the difference of their reciprocals is 1.

SOLUTION. Let $x =$ the smaller number
and $y =$ the larger number.

Then for the sum and difference of their reciprocals we have, respectively,

$$\frac{1}{x} + \frac{1}{y} = 5,$$
$$\frac{1}{x} - \frac{1}{y} = 1.$$

This is not a linear system but may be treated as such in the unknowns $1/x$ and $1/y$. Thus, adding the two equations, we obtain

$$\frac{2}{x} = 6,$$

whence $2 = 6x$ and $x = \frac{1}{3}$.

Subtracting the second equation from the first, we obtain

$$\frac{2}{y} = 4,$$

whence $2 = 4y$ and $y = \frac{1}{2}$.

Hence the two numbers are $\frac{1}{3}$ and $\frac{1}{2}$. It is left as an exercise to the student to check these results.

NOTE. The similarity of the given system to a linear system will be apparent if we let $u = 1/x$ and $v = 1/y$ so that we have

$$u + v = 5,$$
$$u - v = 1.$$

EXERCISES. GROUP 13

In each of Exs. 1–6, solve and check the given system and illustrate the result graphically.

1. $3x - y = 2, \quad 2x + 3y = 5.$
2. $x + 4y = 7, \quad 2x + 3y = 4.$
3. $2x - 3y = 9, \quad 3x + 4y = 5.$
4. $3x + 2y = 0, \quad 2x + 5y = 11.$
5. $9x + 7y = 0, \quad 5x - 9y = 0.$
6. $2x - 11y = 4, \quad 4x + 7y = 8.$

In each of Exs. 7–10, examine the given system for a solution and illustrate the result graphically.

7. $3x + y = 5, \quad 6x + 2y = 7.$
8. $4x - 2y = 4, \quad 2x - y = 2.$
9. $2x - 6y = 2, \quad x - 3y = 3.$
10. $7x + 2y = 1, \quad 21x + 6y = 3.$

In each of Exs. 11–17, solve and check the given system.

11. $\dfrac{1}{x} + \dfrac{2}{y} = 1, \quad \dfrac{2}{x} - \dfrac{1}{y} = -\dfrac{7}{4}.$
12. $\dfrac{2}{x} - \dfrac{3}{2y} = \dfrac{3}{2}, \quad \dfrac{4}{3x} - \dfrac{3}{y} = \dfrac{1}{3}.$
13. $\dfrac{x}{a} + \dfrac{y}{b} = 3, \quad \dfrac{x}{2a} - \dfrac{3y}{b} = -2.$
14. $ax + by = r, \quad cx + dy = s.$
15. $x + y = 7, \quad y + z = 5, \quad x + z = 6.$
16. $4x + 2y - 7z = 3, \quad x - y - 5z = 1, \quad 2x + 4y + z = 3.$
17. $\dfrac{1}{x} + \dfrac{2}{y} = \dfrac{7}{6}, \quad \dfrac{1}{y} + \dfrac{2}{z} = \dfrac{2}{3}, \quad \dfrac{2}{x} + \dfrac{1}{z} = \dfrac{7}{6}.$

18. In the derivation of Theorem 2 (Sec. 4.7), verify the fact that the unique solution satisfies equation (7).

19. In the system of Theorem 2 (Sec. 4.7) show that if c_1 and c_2 are both zero, the system has a solution $x = 0$, $y = 0$. The system is then said to be *homogeneous*.

20. Let $a_1x + b_1 = 0$, $a_1 \neq 0$, and $a_2x + b_2 = 0$, $a_2 \neq 0$, represent any two linear equations in one variable. Show that a necessary and sufficient condition for these two equations to be consistent is $a_1b_2 - a_2b_1 = 0$.

In each of Exs. 21–30, check the results.

21. If the numerator of a given fraction is increased by 1, its value is changed to $\frac{1}{2}$; if the denominator is increased by 1, its value is changed to $\frac{1}{3}$. Find the fraction.

22. A sum of money was divided equally among a certain number of boys. If there had been two more boys, each would have received \$1 less; if there had been two less, each would have received \$2 more. Find the number of boys and the amount received by each.

23. A two-digit number is equal to 8 times the sum of its digits; if the digits are reversed, the new number is 45 less than the original number. Find the original number.

24. The temperature C measured on the centigrade scale is a linear function of the temperature F measured on the Fahrenheit scale and may be represented by the relation $C = aF + b$, where a and b are constants. Determine these constants and hence this relation by using the facts that the freezing point for water is 0°C and 32°F and that the boiling point is 100°C and 212°F.

25. A train ran a certain distance at a constant speed. If that speed had been increased 10 mi per hour, the trip would have taken 1 hour less; if the speed had been decreased 10 mi per hour, the trip would have taken $1\frac{1}{2}$ hours more. Find the distance and the speed of the train.

26. If the width of a rectangular field is increased by 10 yd and its length is decreased by 10 yd, the area is increased by 400 sq. yd. If the width is decreased by 5 yd and the length is increased by 10 yd, the area is decreased by 50 sq. yd. Find the dimensions of the field.

27. A certain straight line is represented by the linear equation $ax + by = 7$, where a and b are constants (Sec. 3.9). Find a and b if the coordinates of two points on the line are (2, 1) and (−1, 3).

28. It is shown in analytic geometry that a circle may be represented by the equation $x^2 + y^2 + Dx + Ey + F = 0$ where D, E, and F are constants. Determine the values of these constants for the circle which passes through the three points whose coordinates are (0, 0), (3, 6), (7, 0).

29. A and B together can do a certain piece of work in $1\frac{5}{7}$ days, A and C together can do it in $1\frac{7}{8}$ days, and B and C together can do it in $2\frac{2}{9}$ days. Find the number of days in which each alone can do the work.

30. The sum of the digits of a three-digit number is 6. If the hundreds' and tens' digits are interchanged, the new number is 90 more than the original number. If the tens' and units' digits are interchanged, the new number is 9 more than the original number. Find the original number.

5

The quadratic function

5.1. INTRODUCTION

We continue our study of the rational integral function of x for the particular case of degree 2. The function is then called the *quadratic function of* x and is usually written in the form

$$ax^2 + bx + c, \qquad a \neq 0,$$

where a, b, and c are constants. This function is of considerable importance and occurs frequently not only in algebra but also in other branches of mathematics, in physics, and in engineering.

5.2. THE QUADRATIC EQUATION IN ONE VARIABLE

If the quadratic function of x is set equal to zero, we have the *quadratic equation in one variable*,

$$(1) \qquad ax^2 + bx + c = 0, \qquad a \neq 0,$$

where a, b, and c are constants. Equation (1) is also known as the *standard form* of the quadratic equation.

By the *solution* of (1), we mean the determination of its roots (Sec. 4.2). Two methods are commonly employed for effecting the solution: one by factoring and the other by means of a formula. Each method is discussed in the following sections.

5.3. SOLUTION BY FACTORING

The first step in the solution of any quadratic equation by any method is to arrange the equation, if necessary, in the standard form,

$$ax^2 + bx + c = 0, \qquad a \neq 0. \tag{1}$$

The left member of (1) is an example of the general trinomial which may be factored into two linear factors (Sec. 2.9, type 4). Since the product of these two factors is equal to zero, each factor may be set equal to zero in accordance with Theorem 19 (Sec. 2.15). Hence the solution of (1) is reduced to the solution of *two equivalent linear equations* (Secs. 4.3, 4.4).

In equation (1), the only restriction on the constants a, b, and c is that $a \neq 0$. Hence either b or c or both may be zero. We will consider these cases first.

If $c = 0$, equation (1) reduces to

$$ax^2 + bx = 0, \tag{2}$$

which readily factors into

$$x(ax + b) = 0,$$

and is equivalent to the two linear equations

$$x = 0, \qquad ax + b = 0,$$

with the solutions 0 and $-b/a$, which are the roots of (2).

Similarly, we may show that if $b = 0$, the roots are $\pm\sqrt{-\dfrac{c}{a}}$, and if $b = c = 0$, both roots are zero.

We now illustrate the general case, $b \neq 0$, $c \neq 0$, by an example.

Example. By factoring, solve the equation

$$2(x + 1)^2 - x = 4. \tag{3}$$

SOLUTION. We first arrange equation (3) in the standard form,

$$2x^2 + 3x - 2 = 0. \tag{4}$$

Factoring the left member of (4), we have

$$(2x - 1)(x + 2) = 0.$$

Equating each linear factor to zero, we have

$$2x - 1 = 0 \qquad \text{and} \qquad x + 2 = 0,$$

whence $x = \frac{1}{2}$ and $x = -2$, respectively. Hence the required roots are $\frac{1}{2}$ and -2.

As previously noted (Sec. 4.3), the solution of an equation should always be checked by substitution in the *original* equation. Thus, for $x = \frac{1}{2}$ in equation (3), we have $2\left(\frac{1}{2} + 1\right)^2 - \frac{1}{2} = \frac{9}{2} - \frac{1}{2} = 4$; and for $x = -2$, we have $2(-2 + 1)^2 - (-2) = 2 + 2 = 4$. Hence both roots check.

5.4. SOLUTION BY FORMULA

If the left member of a quadratic equation in standard form is easily factored, this is the preferable method of solution. But the solution of a quadratic equation may always be effected by a process known as *completing the square*. This method can always be used even when the solution cannot be readily obtained by factoring. We will illustrate the procedure by an example.

Example 1. Solve the equation

$$2x^2 - 2x - 1 = 0.$$

SOLUTION. The left member cannot be factored into linear factors with rational coefficients. We therefore use the method of completing the square as follows:

Transpose the constant term to the right side of the equation, keeping the variable terms on the left side. Then

$$2x^2 - 2x = 1.$$

Next, divide through by 2, the coefficient of x^2. Then

$$x^2 - x = \frac{1}{2}.$$

The left member becomes a perfect square if we add to it the square of half the coefficient of x. Hence, adding $\left(-\frac{1}{2}\right)^2 = \frac{1}{4}$ to both sides, we have

$$x^2 - x + \frac{1}{4} = \frac{3}{4}$$

or

$$\left(x - \frac{1}{2}\right)^2 = \frac{3}{4}.$$

Next, by extracting the square root of each side we obtain

$$x - \frac{1}{2} = \pm \frac{\sqrt{3}}{2},$$

whence

$$x = \frac{1 \pm \sqrt{3}}{2}.$$

Hence the roots are $\frac{1 + \sqrt{3}}{2}$ and $\frac{1 - \sqrt{3}}{2}$, which the student may check by substitution in the original equation.

Since the method of completing the square is a set procedure, we may employ it to obtain the roots of the standard quadratic equation and then use the resulting solution as a formula. Thus, for

$$ax^2 + bx + c = 0, \qquad a \neq 0.$$

if we transpose c to the right member and then divide through by a, we obtain

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

To complete the square, we add $\left(\frac{b}{2a}\right)^2$ to both members. Then

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

or

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Extracting the square root of each side, we obtain

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

or

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

which is known as the *quadratic formula*.

Conversely, we may show by substitution that each of these values of x satisfies the original standard equation. Accordingly, we have

Theorem 1. *The quadratic equation in one variable,*

$$ax^2 + bx + c = 0, \qquad a \neq 0,$$

has the solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 2. Solve the equation

$$\frac{x+1}{x-1} - 1 = \frac{x}{x-2}$$

SOLUTION. We first clear fractions by multiplying both sides of the equation by the L.C.D. $(x - 1)(x - 2)$. Thus

$$x^2 - x - 2 - x^2 + 3x - 2 = x^2 - x.$$

Simplifying and arranging the terms, we obtain the standard form,

$$x^2 - 3x + 4 = 0.$$

Since the left member cannot be factored into linear factors with rational coefficients, we use the formula of Theorem 1 above. Here $a = 1, b = -3, c = 4$, so that

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{3 \pm \sqrt{9 - 16}}{2} = \frac{3 \pm \sqrt{7}i}{2}.$$

Since the only possible extraneous roots are 1 and 2, the required roots are $\frac{3 \pm \sqrt{7}\,i}{2}$, which may be checked for *accuracy* in the original equation.

Many problems may be solved by quadratic equations.

Example 3. A train traverses 300 mi at a uniform rate. If the rate had been 10 mi per hour greater, the time of the trip would have been 1 hour less. Find the rate of the train.

SOLUTION. Let $x =$ the rate of the train in mi per hour.
Time of trip at original rate $= 300/x$ hours.
Time of trip at increased rate $= 300/(x + 10)$ hours.

$$\therefore \frac{300}{x} - \frac{300}{x+10} = 1.$$

Clearing fractions and arranging the terms in standard form, we have

$$x^2 + 10x - 3000 = 0.$$

Factoring, $$(x + 60)(x - 50) = 0,$$

whence $x = -60, 50$.

The value of $x = 50$ satisfies the original equation and meets the requirements of the problem.

The value of $x = -60$ satisfies the original equation but does not meet the requirements of the problem and hence is rejected. A situation such as this soemtimes occurs in the solution of a problem by means of a quadratic equation. Sometimes both values may meet the conditions of the problem and hence there are two answers; in other cases only one value may be acceptable, as in the present problem.

EXERCISES. GROUP 14

In each of Exs. 1–24, solve the given equation by factoring, if possible; otherwise, by formula. Check each root by substitution in the original equation.

1. $x^2 - 3x + 2 = 0.$
2. $x^2 - x - 12 = 0.$
3. $3y^2 + 2y - 1 = 0.$
4. $6z^2 + z - 2 = 0.$
5. $(x - 2)^2 + 2 = x.$
6. $2(x + 1)^2 - 4 = x(x + 3).$
7. $(x - 3)(x + 2) = 6.$
8. $(y + 1)^2 - 3(y + 1) = 4.$
9. $\dfrac{x - 1}{x + 3} + \dfrac{x - 2}{x + 1} = 1.$
10. $\dfrac{3x - 5}{x + 1} = \dfrac{2(x + 4)}{2x - 3}.$
11. $x^2 - 2x - 1 = 0.$
12. $x^2 - 2x + 2 = 0.$
13. $9u^2 - 12u - 1 = 0.$
14. $4v^2 - 12v + 11 = 0.$
15. $2(x + 2)^2 - (x - 1)^2 = 2x + 7.$
16. $(x + 2)(x - 1) = x + 3.$
17. $(x - 5)(x + 1) = 2(x - 2)^2.$
18. $3(x + 1)^2 = (x + 4)^2 - 12.$
19. $z + \dfrac{10}{z} = 6.$
20. $9y - 12 + \dfrac{7}{y} = 0.$
21. $abx^2 - (a^2 + b^2)x + ab = 0.$
22. $x^2 - 2ax + a^2 + b^2 = 0.$
23. $x^2 - 2bx + b^2 - a = 0.$
24. $4x^2 - 4ax + a^2 = b^2.$

25. In the derivation of Theorem 1 (Sec. 5.4), show that each of the roots obtained satisfies the original standard equation.

26. From the equation $x^2 = a^2$ we obtain the equations $\pm x = \pm a$, but ordinarily we simply write $x = \pm a$. Show that the solution is identical in each case.

27. The length of a room exceeds the width by 5 ft and the area of the room is 150 sq ft. Find its dimensions.

28. A and B together can do a piece of work in $1\frac{7}{8}$ hours, and A alone can do it in 2 hours less than B alone. Find the times in which A and B alone can do the work.

29. A tank can be emptied by two pipes in 2 hours. How long would it take each pipe alone to empty the tank if one pipe can do it in 3 hours less time than the other pipe?

30. One leg of a right triangle is 17 in. longer than the other, and the hypotenuse is 25 in. long. Find the lengths of the legs.

31. A bill of \$600 is to be paid by the members of a club, each one to pay an equal share. Had there been 20 more members, the cost for each member would have been \$1 less. Find the number of members.

32. Find two numbers whose sum is 12 and whose product is 35.

33. It is shown in physics that the distance s(ft) traversed by a body falling in a vacuum is given by the relation $s = v_0t + \frac{1}{2}gt^2$ where v_0 is the initial velocity (ft/sec) of the body, t is the time of descent (sec), and g is the constant acceleration of gravity (ft/sec²). Find the time required by a body to fall 100 ft in a vacuum if the initial velocity is 18 ft/sec and g is 32 ft/sec.²

34. Solve the relation of Ex. 33 for t and explain why only one sign may be taken before the radical.

35. The edges of two cubes differ by 2 in. and their volumes differ by 218 cu in. Find the edge of each cube.

5.5. PROPERTIES OF THE QUADRATIC EQUATION

If the roots of the general quadratic equation

$$(1) \qquad ax^2 + bx + c = 0, \qquad a \neq 0,$$

are represented by r_1 and r_2, then from Theorem 1 (Sec. 5.4),

$$(2) \qquad r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

We now consider the nature of these roots when the coefficients of (1) are all real, that is, a, b, and c are all real numbers. It is evident that the roots depend upon the sign of the expression $b^2 - 4ac$ under the radical sign. Thus, if $b^2 - 4ac > 0$, r_1 and r_2 are real and unequal; if $b^2 - 4ac = 0$, r_1 and r_2 are real and equal; and if $b^2 - 4ac < 0$, r_1 and r_2 are complex and unequal. In this last case the two complex roots differ only in the sign before the imaginary term, that is, if one root is of the form $m + ni$, the other root is of the form $m - ni$, where $i = \sqrt{-1}$. Such roots are called *conjugate complex numbers.*

In view of its significance, the expression $b^2 - 4ac$ is appropriately called the *discriminant* of the quadratic equation (1).

We summarize the preceding result in

Theorem 2. *If a, b, and c are all real numbers in the quadratic equation*

$$ax^2 + bx + c = 0, \qquad a \neq 0,$$

so that the discriminant $D = b^2 - 4ac$ is a real number, then if $D > 0$, the roots are real and unequal; if $D = 0$, the roots are real and equal; and if $D < 0$, the roots are conjugate complex numbers.

Corollary. *If a, b, and c are all rational numbers, the roots are rational if and only if D is non-negative and a perfect square.*

NOTE. If the discriminant D is non-negative and not a perfect square, the roots are radical expressions of the form $m + \sqrt{n}$ and $m - \sqrt{n}$ and are called *conjugate quadratic surds.*

Example 1. Determine the nature of the roots of

$$2x^2 + 5x - 3 = 0.$$

SOLUTION. Here the discriminant $b^2 - 4ac = 5^2 - 4(2)(-3) = 25 + 24 = 49 > 0$. Hence, by Theorem 2, the roots are real and unequal. The student may easily verify this by actually finding the roots. Incidentally, this example also illustrates the corollary to Theorem 2.

Since the roots (2) of the general quadratic equation (1) are expressed in terms of the coefficients, the sum and product of the roots may also be so expressed. Thus, for the sum,

$$r_1 + r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{a},$$

and, for the product,

$$r_1 r_2 = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = \frac{4ac}{4a^2} = \frac{c}{a}.$$

We state these results as

Theorem 3. *For the general quadratic equation,*

$$ax^2 + bx + c = 0, \qquad a \neq 0,$$

the sum of the roots is equal to $-b/a$ and the product of the roots is equal to c/a.

Example 2. Find the value of k in the equation $(k + 1)x^2 - (k + 8)x + 10 = 0$, if the sum of the roots is to be $\frac{9}{2}$.

SOLUTION. By Theorem 3, the sum of the roots is equal to minus the quotient of the coefficient of x by the coefficient of x^2. Hence,

$$\frac{k + 8}{k + 1} = \frac{9}{2}, \text{ whence } 2k + 16 = 9k + 9 \text{ and } k = 1.$$

The student should verify this result by actually finding the roots.

Example 3. Find the value of k in the equation $(k - 1)x^2 - 5x + 3k - 7 = 0$, if one root is to be the reciprocal of the other.

SOLUTION. Let r denote one root. Then the other root is $1/r$ and their product is 1. But the product of the roots is also $(3k - 7)/(k - 1)$. Hence $(3k - 7)/(k - 1) = 1$, whence $3k - 7 = k - 1$ and $k = 3$.

The student should verify this result.

We next establish the following very important theorem.

Theorem 4. *If r is a root of the general quadratic equation*

$$ax^2 + bx + c = 0, \qquad a \neq 0,$$

then $x - r$ is a factor of the left member, and conversely.

PROOF. Let $f(x) = ax^2 + bx + c$.

Since r is a root of $f(x) = 0$, we may write

$$f(r) = ar^2 + br + c = 0.$$

By subtraction, we have

$$f(x) - f(r) = ax^2 + bx + c - (ar^2 + br + c)$$

or

$$f(x) - 0 = a(x^2 - r^2) + b(x - r).$$

Hence,

$$f(x) = (x - r)[a(x + r) + b],$$

and $x - r$ is a factor of $f(x)$.

Conversely, if $x - r$ is a factor of $f(x)$, we may write

$$f(x) = (x - r)P(x),$$

where $P(x)$ is the other factor.

Then, for $x = r$, this last relation gives us $f(r) = 0$ by Theorem 19 (Sec. 2.15), and r is a root of $f(x) = 0$.

This completes the proof.

Since r_1 and r_2 as given by (2) are the roots of the general quadratic equation (1), it follows from the preceding theorem that $x - r_1$ and $x - r_2$ are factors of $ax^2 + bx + c$. For the product of these factors we have

$$\begin{aligned}(x - r_1)(x - r_2) &= \left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right) \\ &= \left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a} \\ &= x^2 + \frac{b}{a}x + \frac{c}{a} = \frac{1}{a}(ax^2 + bx + c),\end{aligned}$$

whence we can write the quadratic function in the factored form

$$ax^2 + bx + c = a(x - r_1)(x - r_2). \tag{3}$$

The relation (3) suggests a method for factoring any general trinomial (Sec. 2.9, type 4). We illustrate the process by

Example 4. Factor $6x^2 - 5x - 6$.

SOLUTION. The roots of $6x^2 - 5x - 6 = 0$ are given by the formula as $x = \frac{5 \pm \sqrt{25 + 144}}{12} = \frac{5 \pm 13}{12} = \frac{3}{2}, -\frac{2}{3}$. Hence the factors of $6x^2 - 5x - 6$ are $6\left(x - \frac{3}{2}\right)\left(x + \frac{2}{3}\right) = (2x - 3)(3x + 2)$.

The equation (3) is particularly useful when we wish to determine whether a given quadratic expression is reducible in a particular number field (Sec. 2.8). Since the field is definitely determined by the nature of the roots r_1 and r_2, all that we need do is evaluate the discriminant (Theorem 2).

Example 5. Examine the reducibility of the quadratic expression $x^2 - 2x + 2$.

SOLUTION. The discriminant $b^2 - 4ac = 4 - 4 \cdot 2 = -4$ so that the zeros of the given expression are conjugate complex numbers. Hence the expression is irreducible in the field of real numbers.

We have already seen that the quadratic equation has two roots. We now investigate the possibility of there being more than two roots. Suppose that equation (1) also has the root r different from either r_1 or r_2 as given by (2). Substituting this value of x in (3), we have

$$ar^2 + br + c = a(r - r_1)(r - r_2),$$

where all the factors in the right member are different from zero. Hence, by Theorem 19 (Sec. 2.15),

$$ar^2 + br + c \neq 0,$$

that is, r is not a root of equation (1). We state this result as

Theorem 5. *The quadratic equation*

$$ax^2 + bx + c = 0, \qquad a \neq 0,$$

has only two roots and they are given by the expressions for r_1 and r_2 in (2).

Heretofore we have solved the problem: Given a quadratic equation, determine its roots. We now consider the converse problem: Given the roots of a quadratic equation, determine the equation. The procedure is illustrated in

Example 6. Form the quadratic equation whose roots are $\frac{4}{3}$ and $\frac{3}{4}$.

SOLUTION. The equation may first be expressed in the form

$$\left(x - \frac{4}{3}\right)\left(x - \frac{3}{4}\right) = 0.$$

Expanding,

$$x^2 - \frac{25}{12}x + 1 = 0.$$

Multiplying through by 12 for greater neatness, we have the required equation

$$12x^2 - 25x + 12 = 0.$$

The equation may also be obtained by using the sum and product of the roots.

EXERCISES. GROUP 15

In each of Exs. 1–6, without solving the given equation, determine the nature, sum, and product of the roots.

1. $x^2 + x - 6 = 0.$
2. $x^2 + 4x + 4 = 0.$
3. $x^2 - 2x + 3 = 0.$
4. $(x + 1)^2 = x - 1.$
5. $x + \dfrac{1}{x} = 4.$
6. $\dfrac{x + 1}{x - 1} = \dfrac{3x - 1}{x + 1}.$

In each of Exs. 7–12, determine the value (s) of k for which the given equation has equal roots.

7. $kx^2 + 8x + 4 = 0.$
8. $x^2 - 3kx + 9 = 0.$
9. $x^2 + kx + 8 = k.$
10. $x^2 + 3k + 1 = (k + 2)x.$
11. $(k + 4)x^2 - 1 = (2k + 2)x - k.$
12. $(k - 1)x^2 - 2kx + k^2 = 0.$

In each of Exs. 13–18, form the equation having the indicated roots.

13. 3, 4.
14. $\frac{5}{6}, -\frac{3}{2}.$
15. $\sqrt{2}, -\sqrt{2}.$
16. $1 + i, 1 - i.$
17. $1 + \sqrt{5}, 1 - \sqrt{5}.$
18. $2 + 3i, 2 - 3i.$

In each of Exs. 19–22, examine the reducibility of the given quadratic expression, and find its factors without restriction as to number field.

19. $x^2 - 7x + 10.$
20. $x^2 + 4x + 1.$
21. $x^2 + 2x + 5.$
22. $2x^2 - 2x + 5.$

23. If one root of the equation $x^2 + kx - 2 = 0$ is 1, find the value of k and the other root.

24. Find the value of k in the equation $2kx^2 - (12k + 1)x + 12 = 0$ if the sum of the roots is to be 7.

25. Find the value of k in the equation $(k - 2)x^2 - 5x + 2k = 0$ if the product of the roots is to be 6.

26. If one root of the equation $(k^2 - 3)x^2 - 3(k - 1)x - 5k = 0$ is -2, find the values of k.

27. If one root of the equation $3x^2 + (k - 1)x - 12 = 0$ is the negative of the other, find the value of k.

28. Find the value of k in the equation $(k + 2)x^2 + 10x + 3k = 0$ if one root is the reciprocal of the other.

29. Find the value of k in the equation $x^2 - 3kx + 2k + 1 = 0$ if the roots differ by 4.

30. Find the values of k in the equation $2x^2 - 4x + k^2 - 2k - 3 = 0$ if one root is equal to zero.

31. Find the values of a and b in the equation $x^2 + (2a + 3b - 1)x + a - b - 3 = 0$ if both roots are equal to zero.

32. Find the values of k in the equation $2kx^2 + 3x + k = 0$ if one root is twice the other.

33. Find the value of k in the equation $x^2 + (2k + 5)x + k = 0$ if one root exceeds the other by 3.

34. Establish the Corollary to Theorem 2 (Sec. 5.5).

35. Establish Theorem 4 (Sec. 5.5) by actually dividing $ax^2 + bx + c$ by $x - r$ and then showing that the remainder is identically zero.

36. Prove that a necessary and sufficient condition for a quadratic equation to have a zero root is that the constant term be zero.

37. If one root of $ax^2 + bx + c = 0$ is double the other, show that $2b^2 = 9ac$.

38. If the coefficients of $ax^2 + bx + c = 0$ are real, a and b are both positive, and c is negative, show that one root is positive and the other negative.

39. Show that if the complex number $m + ni$ is a root of the general quadratic equation with real coefficients, then the conjugate complex number $m - ni$ is also a root.

40. Form the quadratic equation with real coefficients if one of its roots is $1 + 2i$ where $i = \sqrt{-1}$.

41. Show that if the quadratic surd $m + \sqrt{n}$ is a root of the general quadratic equation with rational coefficients, then the conjugate quadratic surd $m - \sqrt{n}$ is also a root.

42. Form the quadratic equation with rational coefficients if one of its roots is $1 + \sqrt{2}$.

43. Prove that if the equation $x^2 + bx + c = 0$, where b and c are integers, has rational roots, these roots must be integers.

44. Show that the sum of the reciprocals of the roots of $ax^2 + bx + c = 0$ is equal to $-b/c$.

45. Show that the sum of the squares of the roots of $ax^2 + bx + c = 0$ is equal to $b^2/a^2 - 2c/a$.

5.6. EQUATIONS IN QUADRATIC FORM

Heretofore we have considered the general quadratic equation

(1) $$ax^2 + bx + c = 0, \qquad a \neq 0,$$

where the unknown is simply the variable x. If, however, the unknown is any other function of x, say $f(x)$, then (1) may be written symbolically in the form

(2) $$a[f(x)]^2 + b[f(x)] + c = 0, \qquad a \neq 0,$$

and equation (2) is said to be in *quadratic form*. Evidently the requirement for an equation to be in quadratic form is that only $f(x)$ and its square appear in the equation. Hence, by a suitable substitution, equation (2) may be transformed into the form (1). For example, the equation

$$x^4 - 7x^2 + 12 = 0,$$

being an equation of the fourth degree, is not a quadratic equation, but it is in quadratic form since, if we let $y = x^2$, it may be written as

$$y^2 - 7y + 12 = 0.$$

Solving this equation for the two values of y, we may set x^2 equal to each value, and from these two equations obtain the required four roots of the given equation. The process is shown in

Example 1. Solve: $x^4 - 7x^2 + 12 = 0.$

SOLUTION. Let $y = x^2$ so that the given equation assumes the form

$$y^2 - 7y + 12 = 0.$$

Factoring, $$(y - 4)(y - 3) = 0.$$

Hence, $y = 4$ and $y = 3$ from which we have

$$x^2 = 4, \text{ whence } x = \pm 2,$$

and $$x^2 = 3, \text{ whence } x = \pm\sqrt{3}.$$

These are the four solutions of the given equation.

The student should also solve this equation by factoring immediately.

Example 2. Solve: $\dfrac{3(x^2 + 1)}{x} + \dfrac{2x}{x^2 + 1} = 7.$

SOLUTION. This equation is not in quadratic form as it stands, and if we clear of fractions we obtain a fourth degree equation which is also not in quadratic form. However, we may note that the given equation involves reciprocals, in which case the proper substitution will lead to a quadratic equation. Thus, let

$$y = \frac{x^2 + 1}{x}.$$

Then the given equation becomes

$$3y + \frac{2}{y} = 7.$$

Multiplying by y, $$3y^2 - 7y + 2 = 0.$$

Factoring, $$(y - 2)(3y - 1) = 0,$$

whence $$y = 2, \tfrac{1}{3}.$$

For $$y = 2, \qquad \frac{x^2 + 1}{x} = 2,$$

whence $$x^2 - 2x + 1 = 0,$$

and $$x = 1, 1.$$

For $$y = \frac{1}{3}, \qquad \frac{x^2 + 1}{x} = \frac{1}{3},$$

whence $$3x^2 - x + 3 = 0.$$

By the quadratic formula, $$x = \frac{1 \pm \sqrt{1 - 36}}{6} = \frac{1 \pm \sqrt{35}\, i}{6}.$$

Hence, the required roots are 1, 1, $\dfrac{1 \pm \sqrt{35}\, i}{6}$.

Some equations that contain radicals involving square roots may be in quadratic form. In connection with such equations it is important to note a certain convention regarding the signs before the radicals. It must be understood as *a matter of notation* that, if no sign appears before the indicated square root of a quantity, the *positive square root is always meant.* If the negative square root is intended, the minus sign must appear before the radical. Thus the positive square root of a quantity x is written as $\sqrt{x}$, the negative square root as $-\sqrt{x}$, and *both* the positive and negative square roots as $\pm\sqrt{x}$.

Example 3. $x^2 + 3x - \sqrt{x^2 + 3x - 1} - 7 = 0.$

SOLUTION. To solve this equation we must eliminate the radical. One method is to transpose the radical to the right member and then to square both sides. This, however, leads to a fourth degree equation which is not in quadratic form. Furthermore, the operation of squaring may possibly introduce extraneous roots (Sec. 4.3).

We may, however, proceed as follows. Although we cannot alter the radicand $x^2 + 3x - 1$, we can rearrange the given equation thus:

$$x^2 + 3x - 1 - \sqrt{x^2 + 3x - 1} - 6 = 0.$$

Now let $y = \sqrt{x^2 + 3x - 1}$, the *positive* square root.

Then $$y^2 - y - 6 = 0.$$

Factoring, $$(y - 3)(y + 2) = 0,$$

whence $$y = 3, -2.$$

Now, in accordance with our substitution, y can have only a positive value. Hence,

$$\sqrt{x^2 + 3x - 1} = 3.$$

Squaring, $$x^2 + 3x - 1 = 9,$$

whence $$x^2 + 3x - 10 = 0.$$

Factoring, $$(x - 2)(x + 5) = 0,$$

whence $$x = 2, -5.$$

Since both of these values check the *original* equation, they are the required solutions.

If, contrary to our substitution, we set the radical equal to -2, we would obtain two extraneous solutions.

5.7. RADICAL EQUATIONS

An equation which contains one or more radicals involving the unknown quantity is called a *radical equation*. We consider here radical equations which involve only square roots and whose solutions lead only to linear or quadratic equations. Examples of such equations are $\sqrt{x + 6} + \sqrt{x - 2} - 4 = 0$ and $\sqrt{x^2 - 3x + 4} = 2$.

To solve a radical equation we must eliminate the radicals by rationalizing. The general procedure is to arrange the given equation so that a single radical appears alone on one side. Squaring both sides will then eliminate this radical. This process, known as *isolating the radical*, may then be repeated for each of any remaining radicals.

Example 1. Solve: $\sqrt{x + 6} + \sqrt{x - 2} - 4 = 0.$

SOLUTION. We first isolate one radical, say $\sqrt{x - 2}$, by transposing it to the right member. Thus,

$$\sqrt{x + 6} - 4 = -\sqrt{x - 2}.$$

Squaring, $$x + 6 - 8\sqrt{x + 6} + 16 = x - 2.$$

Isolating the radical and simplifying, $-8\sqrt{x + 6} = -24.$

Dividing by -8, $$\sqrt{x + 6} = 3.$$

Squaring, $$x + 6 = 9,$$

whence $$x = 3.$$

By substitution we find that this value of x satisfies the original equation and is therefore the solution.

NOTE. At one step in the solution above we divided both sides of the equation by -8. Students commonly omit this step; consequently, in the subsequent squaring they have to deal with figures which are often unnecessarily large and awkward to handle. In this particular case the figures would all be 64 times as great as shown.

Since the solution of radical equations involves squaring, it is highly important to check all solutions in the original equation in order to detect any possible extraneous roots (Sec. 4.3). We may note also that some radical equations have no solution whatsoever, as the following example illustrates.

Example 2. Solve $\sqrt{x-3} - \sqrt{2x+2} = 2$.

SOLUTION. Transposing, $\sqrt{x-3} - 2 = \sqrt{2x+2}$.

Squaring, $x - 3 - 4\sqrt{x-3} + 4 = 2x + 2$.

Isolating the radical and simplifying,

$$-4\sqrt{x-3} = x + 1.$$

Squaring, $16x - 48 = x^2 + 2x + 1$.

Transposing, $x^2 - 14x + 49 = 0$.

Solving, $x = 7, 7$.

If we substitute 7 for x in the original equation, we obtain

$$\sqrt{7-3} - \sqrt{14+2} = 2 - 4 \neq 2.$$

Hence the given equation has no solution.

EXERCISES. GROUP 16

In each of Exs. 1–15, solve the given equation as one in the quadratic form.

1. $x^4 - 17x^2 + 16 = 0$.
2. $2x^4 + 17x^2 - 9 = 0$.
3. $x + x^{1/2} - 6 = 0$.
4. $x^{1/2} - 3x^{1/4} + 2 = 0$.
5. $2x^{1/2} + 2x^{-1/2} - 5 = 0$.
6. $x^{1/3} + 2x^{-1/3} - 3 = 0$.
7. $\left(x + \frac{1}{x}\right)^2 + 4\left(x + \frac{1}{x}\right) = 12$.
8. $3\left(\frac{x-1}{x}\right)^2 - 4\left(\frac{x-1}{x}\right) = 4$.
9. $2\frac{x^2-2}{x} - \frac{x}{x^2-2} = 1$.
10. $x^2 - \frac{1}{a^2} = a^2 - \frac{1}{x^2}$.
11. $\sqrt{\frac{x+3}{x-3}} - 2\sqrt{\frac{x-3}{x+3}} = 1$.
12. $2x^2 - 2x + \sqrt{x^2 - x} = 3$.
13. $x^2 + 2x + \sqrt{x^2 + 2x + 10} - 20 = 0$.
14. $2x^2 + 2x - 3\sqrt{x^2 + x + 3} - 3 = 0$.

15. $\dfrac{1+\sqrt{1+x^2}}{x} + \dfrac{x}{1+\sqrt{1+x^2}} - 2\sqrt{2} = 0.$

In each of Exs. 16–23, solve the radical equation and test for extraneous roots.

16. $\sqrt{x+2} + \sqrt{x+7} = 5.$

17. $\sqrt{x+2} - \sqrt{x+7} = 5.$

18. $\sqrt{x^2 - 3x + 4} = 2.$

19. $\sqrt{x+2} + \sqrt{2x+5} = 5.$

20. $\sqrt{1+\sqrt{3+\sqrt{6x}}} = 2.$

21. $\sqrt{x - \sqrt{1-x}} + \sqrt{x} = 1.$

22. $\sqrt{2x-1} - \sqrt{3x+10} + \sqrt{x-1} = 0.$

23. $\sqrt{x+3} + \sqrt{2-x} - \sqrt{x+8} = 0.$

In each of Exs. 24–25, rationalize the given equation, that is, transform it into another equation which is entirely free of radicals.

24. $\sqrt{x} + \sqrt{y} = 1.$

25. $\sqrt{(x-3)^2 + y^2} + \sqrt{(x+3)^2 + y^2} = 10.$

5.8. GRAPH OF THE QUADRATIC FUNCTION

The graph of the quadratic function $ax^2 + bx + c$, $a \neq 0$, is obtained by setting y equal to this function and then computing corresponding real values of x and y from the equation

$$y = ax^2 + bx + c, \qquad a \neq 0.$$

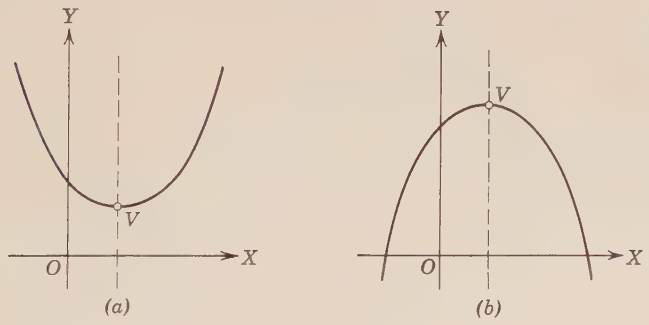


Figure 8

The number pairs are the coordinates of points which are plotted and have a smooth curve drawn through them (Sec. 3.9). The graph has the appearance shown in Fig. 8 and is called a *parabola*. Several facts about the parabola are proved in analytic geometry. Thus, if $a > 0$, the curve opens upward (Fig. 8(a)) and if $a < 0$, the curve opens downward (Fig. 8(b)).

Furthermore, the curve is symmetric with respect to a vertical line called the *axis* of the parabola. The point of intersection V of the axis and the parabola is called the *vertex*. If $a > 0$, V is called the *minimum point* and its ordinate y represents the smallest or *minimum value* of the quadratic function. If $a < 0$, V is called the *maximum point* and its ordinate y represents the largest or *maximum value* of the quadratic function. These facts, proved in analytic geometry, are recorded in

Theorem 6. *The quadratic function*

$$ax^2 + bx + c, \qquad a \neq 0, \tag{1}$$

is represented graphically by the parabola

$$y = ax^2 + bx + c, \tag{2}$$

whose axis is parallel to (*or coincident with*) *the Y-axis, and whose vertex is the point* $(-b/2a, c - b^2/4a)$.

If $a > 0$, *the parabola* (2) *opens upward and its vertex is a minimum point, and the quadratic function* (1) *has a minimum value equal to* $c - b^2/4a$ *when* $x = -b/2a$.

If $a < 0$, *the parabola* (2) *opens downward and its vertex is a maximum point, and the quadratic function* (1) *has a maximum value equal to* $c - b^2/4a$ *when* $x = -b/2a$.

Maximum and minimum values will be discussed further in the next section. We shall now consider the graphical representation of the zeros of a quadratic function and shall do this by means of actual examples.

Example. Draw the graph and discuss the zeros of each of the following functions:

(a) $x^2 - x - 2$. (b) $x^2 - 4x + 4$. (c) $x^2 + x + 2$.

SOLUTION. (a) Set $y = x^2 - x - 2$ and obtain the coordinates of a suitable number of points as shown in the table of Fig. 9. The graph, Fig. 9, crosses the X-axis where $x = -1$ and $x = 2$ and these are the *zeros* of the given function or the *roots* of the equation $x^2 - x - 2 = 0$. The graph also shows that the function is positive for all values of x less than -1 and greater than 2, and is negative for all values of x between -1 and 2.

(b) For $y = x^2 - 4x + 4$ we obtain the table of values shown and the graph given by Fig. 10. Here the graph does not cross the X-axis but touches it at the point where $x = 2$. This is a point of tangency and indicates that while there are two zeros, they are both equal to 2. In other words, the roots of $x^2 - 4x + 4 = 0$ are both equal to 2. Also, the given function is positive for all real value of x except $x = 2$.

(c) From $y = x^2 + x + 2$ we obtain the table of values shown and the graph given by Fig. 11. Here the graph neither crosses nor touches the

x	y
0	−2
1	−2
2	0
3	4
−1	0
−2	4

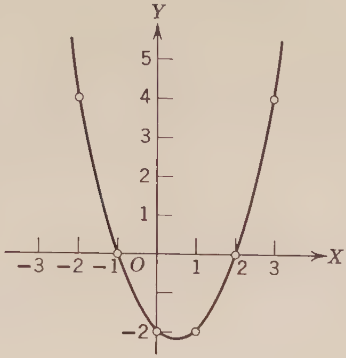


Figure 9

x	y
0	4
1	1
2	0
3	1
4	4
5	9
−1	9

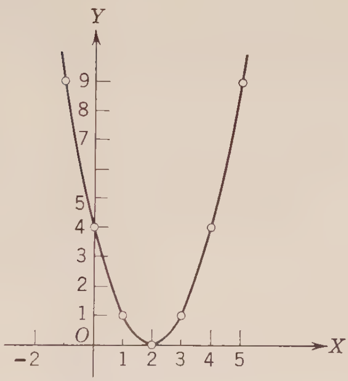


Figure 10

x	y
0	2
1	4
2	8
−1	2
−2	4
−3	8

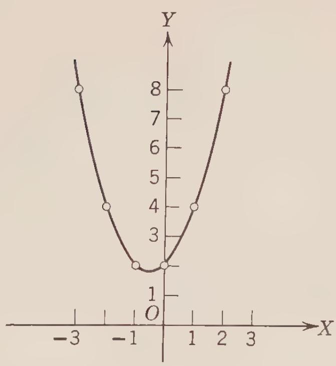


Figure 11

X-axis. Hence there are no *real* zeros. The roots of $x^2 + x + 2 = 0$ are easily found to be the conjugate complex numbers $\dfrac{-1 \pm \sqrt{7}\,i}{2}$. Also, the graph shows that the given function is positive for all real values of x.

5.9. MAXIMA AND MINIMA

We now consider the algebraic determination of the extreme values (maximum and minimum) of the quadratic function $ax^2 + bx + c$, $a \neq 0$, where a, b, c, and x are all assumed to be real numbers.

We first note that the square of any *real* number is either zero or positive. Hence *the minimum value of the square of a real expression is zero.*

Next we transform the quadratic function by completing the square in x. Thus,

$$y = ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x\right) + c$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a},$$

whence

(1) $$y = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}.$$

For any given quadratic function, a, b, and c are *constants* and x is the only variable. Hence the value of y is controlled by the value assigned to x. We now examine the relation (1) for two cases: $a > 0$ and $a < 0$.

$a > 0$. In this case y has no relative (finite) maximum value since it may be made as algebraically large as we please by assigning a sufficiently large numerical value to x. But it does have a minimum value when $\left(x + \dfrac{b}{2a}\right)^2 = 0$ or $x = -b/2a$, and this minimum value is $c - b^2/4a$.

$a < 0$. In this case y has no relative (finite) minimum value since it may be made as algebraically small as we please by assigning a sufficiently large numerical value to x. But it does have a maximum value when $\left(x + \dfrac{b}{2a}\right)^2 = 0$ or $x = -b/2a$, and this maximum value is $c - b^2/4a$.

These results are in agreement with Theorem 6 (Sec. 5.8). We record them as

Theorem 7. *The quadratic function* $ax^2 + bx + c$, $a \neq 0$, *where* a, b, *and* c *are all real constants, has an extreme value equal to* $c - b^2/4a$ *when* $x = -b/2a$. *This extreme value is a minimum when* $a > 0$ *and a maximum when* $a < 0$.

The value of this theorem lies in the fact that it may be used to solve any problem in maxima or minima which leads to a quadratic function in one variable. The general problem of determining maximum and minimum values for any function belongs to the calculus and will not be considered here.

Example 1. Determine the extreme value of the quadratic function $6 + x - x^2$. Illustrate the result graphically.

SOLUTION. Since the coefficient of x^2 is negative, the extreme value is a maximum which may be obtained directly by substitution in the results of Theorem 7. Thus, for $a = -1$, $b = 1$, $c = 6$, the maximum value is $c - \frac{b^2}{4a} = 6 - \frac{1}{-4} = \frac{25}{4}$ when $x = -\frac{b}{2a} = -\frac{1}{-2} = \frac{1}{2}$. However, should the student forget these expressions he can always obtain the results by completing the square in x, as in the derivation of Theorem 7.

The graph of the function is shown in Fig. 12 with its maximum point and zeros.

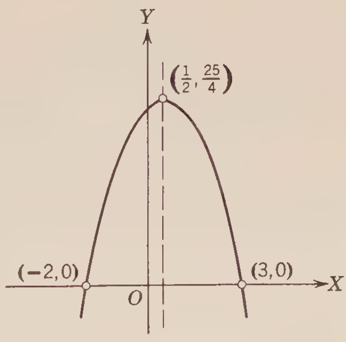


Figure 12

We next consider a typical problem in maxima and minima which leads to a quadratic function.

Example 2. The sum of two numbers is 8. Find these numbers if the sum of their squares is to be a minimum.

SOLUTION. Let x = one number. Then $8 - x$ = the other number.

The general procedure in problems of this type is to express the quantity to be maximized or minimized as a function of a single variable. Thus, if S represents the sum of the squares of these numbers, we write

$$S = x^2 + (8 - x)^2 = 2x^2 - 16x + 64.$$

By Theorem 7, S has a minimum value when $x = -\frac{b}{2a} = -\frac{-16}{4} = 4$. Hence the required numbers are 4 and 4.

As noted in Example 1 above, should the student forget the formulas of Theorem 7 he can always effect the solution by completing the square. The student should illustrate this problem graphically.

EXERCISES. GROUP 17

In each of Exs. 1–6, determine the maximum or minimum value of the given function, and illustrate the result graphically.

1. $4x^2 + 16x + 19$.
2. $24x - 3x^2 - 47$.
3. $x^2 - 6x + 9$.
4. $4x - 2x^2 - 5$.
5. $3 + 2x - x^2$.
6. $3 + 2x + x^2$.

In each of Exs. 7–12, determine the values of x for which the given function is positive, negative, and zero, and has either a maximum or minimum value. Illustrate the results graphically.

7. $x^2 - 5x + 4$.
8. $3 - 5x - 2x^2$.
9. $x^2 - 2x + 1$.
10. $2x - x^2 - 1$.
11. $x^2 - x + 1$.
12. $x - x^2 - 2$.

13. On the same set of coordinate axes plot the three functions $x^2 - x - 6$, $x^2 - x - 1$, $x^2 - x + 4$, and note the effect produced by varying the constant term.

The problems stated in Exs. 14–20 should be illustrated graphically.

14. Divide the number 12 into two parts such that their product shall be a maximum.

15. Find the number which exceeds its square by the greatest amount.

16. The perimeter of a rectangle is 20 in. Find its dimensions if its area is to be a maximum.

17. The sum of the lengths of the legs of a right triangle is constant and equal to 14 in. Find the lengths of the legs if the area of the triangle is to be a maximum.

18. Show that of all rectangles having a fixed perimeter, the one of greatest area is a square.

19. A rectangular lot alongside a river is to be fenced in on three sides by 100 ft of wire, no wire being required for the river side. Find the dimensions of the lot if its area is to be a maximum.

20. A simple beam of length l feet is uniformly loaded with w pounds per foot. It is shown in mechanics that, at a distance of x feet from one support, the bending moment M in pound-feet is given by the formula $M = \frac{1}{2}wlx - \frac{1}{2}wx^2$. Prove that the bending moment has its maximum value at the center of the beam.

In each of Exs. 21–23, let $y = ax^2 + bx + c$ be a quadratic function such that the roots of $y = 0$ are r_1 and r_2.

21. If r_1 and r_2 are real and unequal, and $r_1 > r_2$, show that y has the same sign as a when $x > r_1$ and $x < r_2$, and is opposite in sign to a when $r_1 > x > r_2$.

22. If r_1 and r_2 are real and equal, show that y has the same sign as a when $x \neq r_1$.

23. If r_1 and r_2 are conjugate complex numbers, show that y has the same sign as a for all values of x.

24. Find the expression for the family or set of quadratic functions of x each of which has a maximum value of 4 when $x = -2$.

25. Find the expression for the family or set of quadratic functions of x each of which has a minimum value of 5 when $x = 3$.

5.10. THE QUADRATIC EQUATION IN TWO VARIABLES

The general equation of the second degree in two variables x and y is represented by

$$ax^2 + bxy + cy^2 + dx + ey + f = 0, \tag{1}$$

where the coefficients a, b, c, d, e, and f are constants with the restriction that at least one of the three coefficients a, b, and c is different from zero.

Since equation (1) is a relation between the variables x and y, it will, in general, have a graphical representation (Sec. 3.9). It is shown in analytic geometry that the graph of equation (1), if it exists in real coordinates, is either a curve known as a *conic section* or else a limiting case which may be a point, a single line, or a pair of lines.

The type of conic section represented by (1) depends upon the coefficients. In order to obtain the graph and properties of this curve most easily, the equation is usually transformed into a simpler form. We list here some of these simplified equations together with their graphs.

The circle

The equation $x^2 + y^2 = r^2$ represents a *circle* with its center at the origin and having a radius r (Fig. 13).

The parabola

The equation $x = ay^2 + by + c$, $a \neq 0$, represents a *parabola* (Fig. 14) whose axis is horizontal and which opens to the right if $a > 0$ and to the left if $a < 0$. The point V is the *vertex*.

We have previously noted (Sec. 5.8) that the equation $y = ax^2 + bx + c$, $a \neq 0$, represents a parabola whose axis is vertical and which opens upward if $a > 0$ and downward if $a < 0$.

The ellipse

The equation $ax^2 + by^2 = c$, where a, b, and c are all positive, represents an *ellipse* (Fig. 15). In the particular case where $a = b$, the equation represents a circle.

The hyperbola

The equation $ax^2 - by^2 = c$, where a and b are positive and $c \neq 0$, represents a *hyperbola* (Fig. 16).

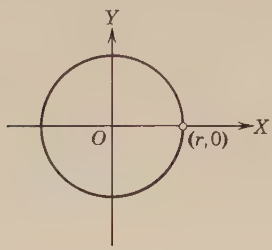


Figure 13

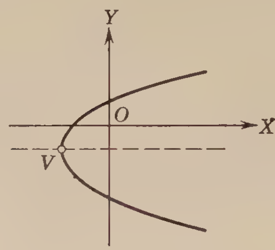


Figure 14

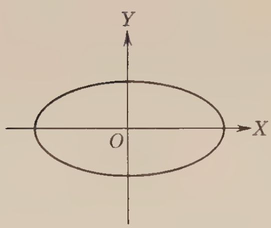


Figure 15

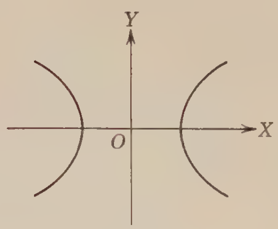


Figure 16

Each of the four curves described above may be obtained as a plane section of a right circular cone. For quadratic equations in two variables, which are different in form from the types listed above, the graphs will be similar in appearance but differently situated with respect to the coordinate axes.

5.11. SYSTEMS OF EQUATIONS INVOLVING QUADRATICS

We now consider a system of two equations in two variables involving quadratics in much the same manner that we discussed a system of two linear equations in two variables (Sec. 4.7). Consider, therefore, the system of two quadratic equations

(1) $$a_1x^2 + b_1xy + c_1y^2 + d_1x + e_1y + f_1 = 0,$$

(2) $$a_2x^2 + b_2xy + c_2y^2 + d_2x + e_2y + f_2 = 0,$$

where the coefficients have the same significance as specified for equation (1) in Sec. 5.10. A common solution of this system may be obtained by eliminating one variable, say y, and then solving for x. Thus, we may solve equation (1) for y in terms of x by means of the quadratic formula, treating x as if it were a constant. If we substitute this value of y in equation (2) and rationalize the result, we obtain, in general, an equation of the fourth degree in x which, as we shall see in a later chapter, has four solutions.

Up to this point, however, we have not considered the general solution of equations of the fourth degree. Hence, in this chapter we will restrict our discussion to certain systems of special types whose complete solution may be effected by solving only linear and quadratic equations. These types are discussed in the sections immediately following.

5.12. SYSTEM HAVING ONE LINEAR EQUATION

If one equation is linear and the other quadratic in a given system, the solution may be effected by substituting from the linear into the quadratic equation. This operation is often performed in mathematics and may be described as *substituting the simpler into the more complicated relation.*

Example. Solve the system

(1) $$x - y = 2,$$

(2) $$x^2 + y^2 = 4,$$

and illustrate the results graphically.

SOLUTION. From equation (1), $y = x - 2$. Substituting this value of y in (2), we have

$$x^2 + x^2 - 4x + 4 = 4,$$

whence $$2x^2 - 4x = 0,$$

and $$x(x - 2) = 0,$$

so that the roots are $x = 0, 2$. The corresponding values of y are obtained from (1). Thus, for $x = 0$, $y = -2$, and for $x = 2$, $y = 0$. We see then that the system has two common solutions; they are $x = 0$, $y = -2$ and $x = 2$, $y = 0$. Each solution should be checked by substitution in *each* of the given equations.

The graph of equation (1) is a straight line; the graph of equation (2) is a circle of radius 2 with its center at the origin. These graphs are shown in Fig. 17. A real solution of an equation in two variables represents the coordinates of a point on the graph of the equation (Sec. 3.9). Hence a *common* real solution of two equations represents the coordinates of a point on the graphs of *both* equations and therefore must represent the *coordinates of their point of intersection*. The common solutions therefore give the two points of intersection, $(0, -2)$ and $(2, 0)$, as shown in Fig. 17.

We next consider the case in which we do not have two *distinct* real common solutions, as in the preceding example. Suppose, now, that our system consists of equation (2) above and the linear equation

$$x - y + 2\sqrt{2} = 0. \tag{3}$$

By the previous process we find that while this system has two solutions, they are both *equal*, namely, $x = -\sqrt{2}$, $y = \sqrt{2}$. Hence there is only *one* point of intersection of the graphs of (2) and (3); it is called a *point of tangency* with the coordinates $(-\sqrt{2}, \sqrt{2})$ as shown in Fig. 17, and the line (3) is said to be *tangent* to the circle (2).

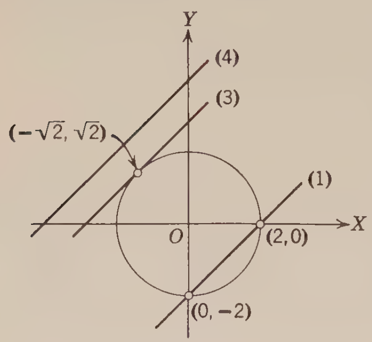


Figure 17

Finally, let us consider the system consisting of equation (2) and the linear equation

$$x - y + 4 = 0. \tag{4}$$

The common solutions of this system are found to be $x = 2 + \sqrt{2}\,i$, $y = -2 + \sqrt{2}\,i$ and $x = 2 - \sqrt{2}\,i$, $y = -2 - \sqrt{2}\,i$. These solutions are both complex and, since only real coordinates can be plotted, it means that the line (4) and the circle (2) do not intersect, as shown in Fig. 17.

We have thus illustrated the algebraic and geometric analogs for a system consisting of a quadratic and a linear equation in two variables.

NOTE. In obtaining the solutions of a system of equations involving quadratics the student must be careful to pair off the values properly. Interchanging values will give incorrect solutions which may always be detected by checking in the *original* system.

5.13. SYSTEM OF EQUATIONS OF THE FORM $ax^2 + by^2 = c$

If each equation of the system is of the form $ax^2 + by^2 = c$, the system should first be solved as a linear system in x^2 and y^2 (Sec. 4.7). The required values of x and y may then be obtained by the simple extraction of square roots.

Example. Solve the system

(1) $x^2 + 4y^2 = 8,$

(2) $2x^2 - y^2 = 7,$

and illustrate the results graphically.

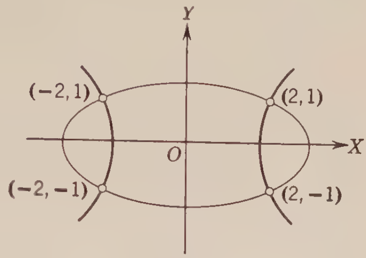


Figure 18

SOLUTION. We first solve the given system for x^2 and y^2. Thus, multiplying (2) by 4, we have

(3) $8x^2 - 4y^2 = 28.$

Adding (1) and (3), $9x^2 = 36,$

whence $x^2 = 4$ and $x = \pm 2.$

Substituting this value of x^2 in (2), we have $y^2 = 1$ and $y = \pm 1$.

The student must be careful to note that there are actually *four* and not two solutions, for each value of x may be paired with both values of y. Thus the four solutions may be shown as paired in the following table:

x	2	2	−2	−2
y	1	−1	1	−1

The solutions are shown graphically in Fig. 18.

EXERCISES. GROUP 18

In each of Exs. 1–10, solve the given system and illustrate the results graphically.

1. $2x - y = 6,$
   $y^2 = x.$

2. $x + y = 2,$
   $x^2 + y^2 = 4.$

3. $2x + y = 4,$
   $y^2 - 4x = 0.$

4. $3x - y - 8 = 0,$
   $x^2 + y^2 - 4x - 6y + 8 = 0.$

5. $2x - 3y = 5,$
   $2x^2 + 3y^2 = 5.$

6. $x - y + 2 = 0,$
   $y^2 - 8x = 0.$

7. $x + y = 5,$
   $x^2 + y^2 = 9.$

8. $2x - y + 2 = 0,$
   $y^2 = 4x.$

9. $x - y = 0,$
   $2x^2 - xy + 2y^2 = 3.$

10. $x + y = 1,$
    $x^2 - 2xy + y^2 - 2x - 2y + 1 = 0.$

11. Find the values k must assume in order that the straight line $y = x + k$ may be tangent to the circle $x^2 + y^2 - 10x + 2y + 18 = 0.$

12. Find the value k must assume in order that the straight line $x + y = k$ may be tangent to the parabola $y^2 = 8x.$

In each of Exs. 13–20, solve the given system and illustrate the results graphically.

13. $x^2 + y^2 = 4,$
$4y^2 - x^2 = 4.$

14. $x^2 - y^2 = 5,$
$9x^2 + 16y^2 = 145.$

15. $4x^2 + 9y^2 = 36,$
$9x^2 + 4y^2 = 36.$

16. $x^2 + 4y^2 = 16,$
$x^2 + y^2 = 9.$

17. $x^2 + y^2 = 16,$
$9x^2 + 16y^2 = 144.$

18. $x^2 + y^2 = 2,$
$2y^2 - x^2 = 4.$

19. $x^2 + y^2 = 1,$
$x^2 - y^2 = 4.$

20. $x^2 + y^2 = 1,$
$x^2 + y^2 = 4.$

21. Find two positive numbers such that the sum of their squares is equal to 29 and the difference of their squares is equal to 21.

22. The perimeter of a rectangle is 34 ft and the diagonal is 13 ft long. Find the dimensions of the rectangle.

23. Find the dimensions of a rectangle if its perimeter is 80 ft and its area is 375 sq ft.

24. Find the values of k in terms of m and r if the straight line $y = mx + k$ is to be tangent to the circle $x^2 + y^2 = r^2$.

25. Find the value of k in terms of p and m if the straight line $y = mx + k$ is to be tangent to the parabola $y^2 = 4px$.

5.14. SYSTEM OF EQUATIONS OF THE FORM $ax^2 + bxy + cy^2 = d$

If both equations lack terms of the first degree, the solution may be effected by either of two methods which are illustrated in the following

Example. Solve the system

$$x^2 - xy + y^2 = 3, \tag{1}$$

$$x^2 + 2xy - y^2 = 1, \tag{2}$$

and illustrate the results graphically.

SOLUTION. *Method* 1. *Elimination of the constant term.* To eliminate the constant term we multiply equation (2) through by 3, obtaining

$$3x^2 + 6xy - 3y^2 = 3. \tag{3}$$

Subtracting equation (1) from (3), member by member, we have

$$2x^2 + 7xy - 4y^2 = 0,$$

which is free of a constant term and may be factored thus,

$$(2x - y)(x + 4y) = 0,$$

whence we have the two linear relations,

(4) $$2x - y = 0 \quad \text{or} \quad y = 2x,$$

(5) $$x + 4y = 0 \quad \text{or} \quad y = -\frac{x}{4}.$$

We have thus reduced the given system to two simpler systems, each of which has a linear equation (Sec. 5.12). Thus, solving equation (4) with either of the equations (1) or (2), we find $x = \pm 1$ and hence the corresponding values of y are given by $y = 2x = \pm 2$. Similarly, solving (5)

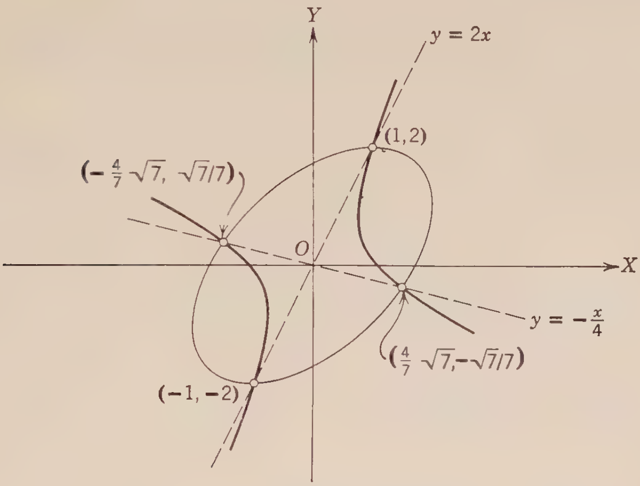


Figure 19

with either of the equations (1) or (2), we find $x = \pm\frac{4}{7}\sqrt{7}$ and hence the corresponding values of y are given by $y = -\dfrac{x}{4} = \mp\dfrac{\sqrt{7}}{7}$. Accordingly, the four required solutions are given by (1, 2), (−1, −2), $\left(\dfrac{4}{7}\sqrt{7}, -\dfrac{\sqrt{7}}{7}\right)$, $\left(-\dfrac{4}{7}\sqrt{7}, \dfrac{\sqrt{7}}{7}\right)$. These solutions are shown graphically in Fig. 19 where the ellipse is the graph of (1) and the hyperbola is the graph of (2).

Method 2. *Use of the substitution* $y = vx$. If we make the substitution $y = vx$ in both equations (1) and (2), we obtain, respectively,

(6) $$x^2 - vx^2 + v^2x^2 = 3 \quad \text{or} \quad x^2 = \frac{3}{1 - v + v^2},$$

(7) $$x^2 + 2vx^2 - v^2x^2 = 1 \quad \text{or} \quad x^2 = \frac{1}{1 + 2v - v^2}.$$

Equating these values of x^2, we have

$$\frac{3}{1 - v + v^2} = \frac{1}{1 + 2v - v^2},$$

whence $$3 + 6v - 3v^2 = 1 - v + v^2,$$

and $$4v^2 - 7v - 2 = 0$$

whose solutions are $v = 2$ and $v = -\frac{1}{4}$.

If we substitute $v = 2$ in either of the relations (6) or (7) we find $x^2 = 1$ whence $x = \pm 1$, and the corresponding values of y are given by $y = vx = 2x = \pm 2$. Similarly, if we substitute $v = -\frac{1}{4}$ in either of the relations (6) or (7), we find $x^2 = \frac{16}{7}$ whence $x = \pm\frac{4}{7}\sqrt{7}$, and the corresponding values of y are given by $y = vx = -\dfrac{x}{4} = \mp\dfrac{\sqrt{7}}{7}$.

These results are in agreement with those obtained by Method 1. The geometric significance of the substitution $y = vx$ is illustrated in Fig. 19 by the dotted lines whose equations are $y = 2x$ and $y = -\dfrac{x}{4}$.

NOTE. If either equation of the system has a constant term equal to zero, that equation may be factored immediately as in Method 1.

5.15. OTHER SYSTEMS

There are other systems of equations whose solutions may be obtained by the solution of a quadratic. Some of these systems are shown in the examples of this article.

An equation in the two variables x and y is said to be *symmetric* with respect to those variables if the equation remains unchanged when x and y are interchanged. Examples of such equations are $x + y = 3$ and $x^2 + xy + y^2 = 7$. A system of two equations, both of which are symmetric with respect to x and y, may be solved by means of a substitution as shown in

Example 1. Solve the system

(1) $$x^2 + y^2 - x - y = 2,$$

(2) $$xy + x + y = 5.$$

SOLUTION. If we make the substitutions $x = u + v$ and $y = u - v$ in equations (1) and (2), we obtain, respectively,

$$(u + v)^2 + (u - v)^2 - (u + v) - (u - v) = 2,$$
$$(u + v)(u - v) + (u + v) + (u - v) = 5.$$

Upon simplification, these equations reduce, respectively, to

(3) $$u^2 + v^2 - u = 1,$$

(4) $$u^2 - v^2 + 2u = 5.$$

By adding these two equations and eliminating v^2, we obtain

$$2u^2 + u - 6 = 0$$

whose solutions are $$u = -2, \tfrac{3}{2}.$$

Substituting $u = -2$ in either equation (3) or (4) we obtain $v = \pm\sqrt{5}\,i$, and for $u = \frac{3}{2}$ we obtain $v = \pm\frac{1}{2}$. The four solutions are then conveniently obtained by the following tabular arrangement.

u	-2	-2	$\frac{3}{2}$	$\frac{3}{2}$
v	$\sqrt{5}i$	$-\sqrt{5}i$	$\frac{1}{2}$	$-\frac{1}{2}$
$x = u + v$	$-2 + \sqrt{5}i$	$-2 - \sqrt{5}i$	2	1
$y = u - v$	$-2 - \sqrt{5}i$	$-2 + \sqrt{5}i$	1	2

We may also effect the solution of some systems in which an equation of degree higher than two appears. This is illustrated in

Example 2. Solve the system

(5) $$x^3 + y^3 = 9,$$

(6) $$x^2 - xy + y^2 = 3.$$

SOLUTION. In this system we note that equation (5) is exactly divisible by (6), giving us the relation

(7) $$x + y = 3.$$

As in Sec. 5.12, we may complete the solution by solving either the system of equations (5) and (7) or the system of equations (6) and (7). The student should show that in either case the solutions are (1, 2) and (2, 1).

It is sometimes possible to solve a system by more than one method. This is illustrated in

Example 3. Solve the system

(8) $$x^2 + y^2 = 13,$$

(9) $$xy = -6.$$

SOLUTION. This system may be solved by the methods of Sec. 5.14 and also by the method of Example 1 above as a symmetric system. We now consider another method.

If we multiply equation (9) by 2 and add the result to and subtract the result from equation (8), we obtain, respectively, the relations

$$x^2 + 2xy + y^2 = 1,$$
$$x^2 - 2xy + y^2 = 25.$$

Extracting the square root of both sides of each equation, we have, respectively,

$$x + y = \pm 1,$$
$$x - y = \pm 5.$$

Using all possible combinations of signs, we have here four linear equations from which, by addition and subtraction, we have, respectively,

$$2x = 6, -6, -4, 4 \quad \text{whence} \quad x = 3, -3, -2, 2,$$
$$2y = -4, 4, 6, -6 \quad \text{whence} \quad y = -2, 2, 3, -3.$$

Hence the solutions are $(3, -2)$, $(-3, 2)$, $(-2, 3)$, $(2, -3)$.

EXERCISES. GROUP 19

In each of Exs. 1–6, solve the given system by either method of Sec. 5.14, and illustrate the results graphically.

1. $x^2 + y^2 = 5,$ $xy = 2.$
2. $x^2 - y^2 = 8,$ $xy = 3.$
3. $x^2 + y^2 = 8,$ $x^2 - xy + 2y^2 = 16.$
4. $xy + 4y^2 = 8,$ $x^2 + 3xy = 28.$
5. $y^2 - x^2 = 16,$ $2y^2 - 4xy + 3x^2 = 17.$
6. $x^2 + xy + y^2 = 7,$ $x^2 - xy - y^2 = 11.$
7. In Method 1 of the example of Sec. 5.14, explain why the results are the same whether we solve each of the linear equations (4) and (5) with either of the given equations (1) or (2).
8. In Method 2 of the example of Sec. 5.14, explain why the results are the same whether we substitute the values of v in *either* of the relations (6) or (7).
9. Solve Example 2 by the method of Example 1 of Sec. 5.15.
10. Solve Example 3 of Sec. 5.15 by the methods of Sec. 5.14.
11. Solve Example 3 of Sec. 5.15 by the method of Example 1 of Sec. 5.15.
12. Solve Ex. 1 by the method of Example 1 of Sec. 5.15.
13. Solve Ex. 1 by the method of Example 3 of Sec. 5.15.

In each of Exs. 14–17, solve the given system by the method of Example 1 of Sec. 5.15, and check the results.

14. $x^2 + y^2 + 2x + 2y = 23,$
$xy = 6.$

15. $x^2 + y^2 + xy = 7,$
$x^2 + y^2 - xy = 3.$

16. $x^2 + y^2 - 2x - 2y = 14,$
$xy + x + y + 5 = 0.$

17. $x^4 + y^4 = 17,$
$x + y = 1.$

18. Solve Ex. 15 by the methods of Sec. 5.14.

In each of Exs. 19–27, solve the given system by any method, and check the results.

19. $x^2 + y^2 = 25,$
$xy = -12.$

20. $x^3 + y^3 = 28,$
$x + y = 4.$

21. $x^3 - y^3 = 56,$
$x^2 + xy + y^2 = 28.$

22. $x^3 + y^3 = 126,$
$x^2 - xy + y^2 = 21.$

23. $2y^2 - xy - x^2 = 44,$
$xy + 3y^2 = 80.$

24. $x^3 + y^3 = 9xy,$
$x + y = 6.$

25. $x^2 + y^2 = 40x^2y^2,$
$x + y = 8xy.$

26. $x^2 + x = 6y,$
$x^3 + 1 = 9y.$

27. $x^2 + y^2 - 4x - 6y + 8 = 0,$
$3x^2 + 3y^2 + 12x - 16y - 10 = 0.$

28. Find two positive numbers whose sum, added to their product, is 34, and the sum of whose squares diminished by their sum is 42.

29. Find two positive numbers whose sum is equal to their product and whose sum added to the sum of their squares is 12.

30. *A* and *B* run a mile race, *B* winning by 1 minute. They run the same race a second time, *A* increasing his speed by 2 mi an hour and *B* decreasing his speed by the same amount. *A* wins by 1 minute. Find their original rates.

6

Inequalities

6.1. INTRODUCTION

We first discussed the concept of inequality in connection with the operation of subtraction and the introduction of negative numbers (Sec. 2.4). We have had little occasion, otherwise, to refer to unequal numbers and inequalities. Now it is our purpose to make a formal study of inequalities and their properties in this chapter.

The subject of inequalities is of considerable importance in many parts of algebra, and we shall see instances of this in our later work. We shall also observe certain analogies between equalities and inequalities.

In determining the relative magnitude of numbers we are said to be *ordering* such numbers. *The order relation is restricted to real numbers* and is illustrated geometrically in the linear coordinate system (Sec. 3.7). In other words, *all of our work on inequalities will apply only to real numbers.* We do not speak of one complex number being greater or less than another.

Previously we have given some definitions of terms and symbols pertaining to inequalities. For convenience, however, these definitions will be repeated in the next section.

6.2. DEFINITIONS AND FUNDAMENTAL THEOREMS

We have already defined an equation as a statement of equality between two expressions (Sec. 4.2). If two expressions are unequal, we have an *inequality*, and one expression is then said to be greater or less than the other.

The real number x is said to be *greater* than the real number y provided that $x - y$ is a positive number. We then write $x > y$, which is read "x is greater than y." Thus, $2 > -3$, for $2 - (-3) = 5$, a positive number.

It follows from this definition that the real number y is *less* than the real number x provided that $y - x$ is a negative number. We then write $y < x$, which is read "y is less than x." Thus, $5 < 7$, for $5 - 7 = -2$, a negative number.

The student should note that, for either inequality symbol, the larger quantity is always at the opening of the symbol which is pointed toward the smaller quantity. We may also note two other useful symbols: $a \geq b$, which is read "a is greater than or equal to b," and $c \leq d$, which is read "c is less than or equal to d." In particular, the inequality $a \geq 0$ is a convenient way of stating that a represents all non-negative numbers.

Two inequalities are said to have the *same sense* if their symbols point in the same direction; otherwise they have the *opposite sense*. Thus the inequalities $a > b$ and $c > d$ have the same sense, but the inequalities $a > b$ and $c < d$ have the opposite sense.

We previously called attention to two types of equations, the identical equation and the conditional equation (Sec. 4.2). Similarly, there are two types of inequalities, the *absolute inequality* and the *conditional inequality*.

An *absolute* or *unconditional inequality* is one that has the same sense for all values for which its members are defined. Examples of absolute inequalities are $5 > -7$ and $x^2 + 1 > 0$.

A *conditional inequality* is one that has the same sense only for certain of the values for which its members are defined. Examples of conditional inequalities are

$$x - 2 < 3, \text{ true only if } x < 5;$$

$$x^2 > 4, \text{ true only if } x > 2 \text{ or if } x < -2.$$

Absolute and conditional inequalities will be discussed in subsequent sections. We will now establish some of the fundamental properties of inequalities.

Theorem 1. *The sense of an inequality remains unchanged if the same quantity is added to or subtracted from both sides, that is, if $a > b$, then $a \pm c > b \pm c$.*

PROOF. By the definition of $a > b$, we have

$$a - b = p, \text{ a positive number,}$$

whence

$$a + c - (b + c) = p,$$

from which, by the definition of "greater than,"

$$a + c > b + c.$$

Similarly, it may be shown that

$$a - c > b - c.$$

Corollary 1. *Any term may be transposed from one side of an inequality to the other side by merely changing its sign.*

In view of Corollary 1, we may transpose every term of an inequality to one side. As a consequence we have

Corollary 2. *Every inequality may be reduced to either of the forms* $A > 0$ *or* $A < 0$, *where* A *is some algebraic expression.*

In view of Theorem 6 (Sec. 2.4), the importance of Corollary 2 lies in the fact that *the solution of any inequality may be reduced to the determination of merely the sign (and not the magnitude) of some expression.*

Theorem 2. *The sense of an inequality remains unchanged if both sides are multiplied or divided by the same positive quantity, that is, if* $a > b$ *and* $c > 0$, *then* $ac > bc$ *and* $a/c > b/c$.

PROOF. From $a > b$, we have

$$a - b = p, \text{ a positive number.}$$

Multiplying both sides by c, we have

$$ac - bc = pc, \text{ a positive number,}$$

whence

$$ac > bc.$$

Similarly, it may be shown that

$$\frac{a}{c} > \frac{b}{c}.$$

By a proof similar to that of Theorem 2, we may establish

Theorem 3. *The sense of an inequality is reversed if both sides are multiplied or divided by the same negative quantity, that is, if* $a > b$ *and* $c < 0$, *then* $ac < bc$ *and* $a/c < b/c$.

Theorem 4. *If two inequalities have the same sense, the corresponding sides may be added together and the sums will be unequal in the same sense, that is, if* $a > b$ *and* $c > d$, *then* $a + c > b + d$.

PROOF. From $a > b$, $a - b = p$, a positive number.
From $c > d$, $c - d = q$, a positive number.
Adding, $a + c - (b + d) = p + q$, a positive number,

whence

$$a + c > b + d.$$

Corollary. *If* $a_1 > b_1, a_2 > b_2, a_3 > b_3, \cdots, a_n > b_n$, *then*

$$a_1 + a_2 + a_3 + \cdots + a_n > b_1 + b_2 + b_3 + \cdots + b_n.$$

Theorem 5. *If of three quantities, the first is greater than the second and the second is greater than the third, then the first is greater than the third, that is, if* $a > b$ *and* $b > c$, *then* $a > c$.

The proof of this theorem is similar to that of Theorem 4 and is left to the student as an exercise.

Theorem 6. *If two inequalities with positive numbers have the same sense, the corresponding sides may be multiplied together, and the products will be unequal in the same sense, that is, if* a, b, c, *and* d *are all positive and* $a > b$ *and* $c > d$, *then* $ac > bd$.

PROOF. Since $c > 0$ and $a > b$, it follows from Theorem 2 that

(1) $$ac > bc.$$

Similarly, since $b > 0$ and $c > d$,

(2) $$bc > bd.$$

From (1), (2), and Theorem 5, we have

$$ac > bd.$$

Corollary 1. *If all quantities are positive and* $a_1 > b_1$, $a_2 > b_2$, $a_3 > b_3, \cdots, a_n > b_n$, *then* $a_1a_2a_3 \cdots a_n > b_1b_2b_3 \cdots b_n$.

Corollary 2. *If* a *and* b *are both positive,* $a > b$, *and* n *is a positive integer, then* $a^n > b^n$.

Corollary 3. *If* a *and* b *are both positive,* $a > b$, *and* n *is a positive integer, then* $a^{1/n} > b^{1/n}$ *(principal roots).*

Corollary 4. *If* a *and* b *are both positive,* $a > b$, *and* n *is a positive integer, then* $a^{-n} < b^{-n}$.

EXERCISES. GROUP 20

1. Complete the proof of Theorem 1 (Sec. 6.2) by showing that if $a > b$, then $a - c > b - c$.
2. Establish Corollary 1 of Theorem 1 (Sec. 6.2).
3. Establish Corollary 2 of Theorem 1 (Sec. 6.2).
4. Complete the proof of Theorem 2 (Sec. 6.2) by showing that if $a > b$ and $c > 0$, then $a/c > b/c$.
5. Establish Theorem 3 (Sec. 6.2).
6. Establish the corollary to Theorem 4 (Sec. 6.2).
7. Show by means of examples that if a, b, c, and d are all positive and $a > b$ and $c > d$, it does not necessarily follow that $a - c > b - d$.
8. Establish Theorem 5 (Sec. 6.2).
9. If $a > b$, $b > c$, and $c > d$, show that $a > d$.

10. If $a > bc$, $c > d$, and $b > 0$, show that $a > bd$.

11. If $a < b$ and $b < c$, show that $a < c$.

12. Establish Corollary 1 of Theorem 6 (Sec. 6.2).

13. Establish Corollary 2 of Theorem 6 (Sec. 6.2).

14. Establish Corollary 3 of Theorem 6 (Sec. 6.2).

15. Establish Corollary 4 of Theorem 6 (Sec. 6.2).

16. Show by means of examples that the result of Theorem 6 does not necessarily follow if a, b, c, and d are not all positive.

17. Show by means of examples that if a, b, c, and d are all positive and $a > b$ and $c > d$, it does not necessarily follow that $a/c > b/d$.

18. If each of two quantities is greater than unity, prove that their product is greater than unity.

19. Using the result of Ex. 18, establish Theorem 6 (Sec. 6.2).

20. If a and b are positive and $a > b$, it follows by Corollary 2 of Theorem 6 (Sec. 6.2) that $a^2 > b^2$. State and prove the converse of this result.

6.3. ABSOLUTE INEQUALITIES

As we have indicated, an absolute inequality is analogous to an identity. We establish its validity by an analytic proof, using one or more of the fundamental principles discussed in Sec. 6.2.

The direct proof of an absolute inequality starts with some known relation and then proceeds by logical steps to the final form desired. Sometimes, however, it is not obvious just what known relation should be used at the beginning. Then it may be possible to make an *analysis* of the desired relation by simplifying it until we obtain a known relation. The direct *proof* is then made by reversing the steps of the analysis. This procedure is illustrated in

Example 1. If a and b are unequal positive numbers, show that

$$a^3 + b^3 > a^2b + ab^2.$$

SOLUTION. Since it does not appear obvious where to start, we attempt to simplify the desired relation in the following

ANALYSIS. We first factor the right member and write

$$a^3 + b^3 > ab(a + b).$$

Since a and b are both positive, $a + b$ is positive and, by Theorem 2 (Sec. 6.2), we may divide both sides by $a + b$ without changing the sense. This gives us

$$a^2 - ab + b^2 > ab.$$

Transposing ab to the left side (Cor. 1, Theorem 1, Sec. 6.2), we have

$$a^2 - 2ab + b^2 > 0,$$

or

$$(a - b)^2 > 0.$$

Now we know this last relation is true, for, $a \neq b$, whence $a - b \neq 0$ and $(a - b)^2 > 0$. Hence this is the starting relation for our

PROOF.

$$(a - b)^2 > 0,$$

whence

$$a^2 - 2ab + b^2 > 0.$$

Transposing $-ab$ to the right side (Cor. 1, Theorem 1, Sec. 6.2), we have

$$a^2 - ab + b^2 > ab.$$

Multiplying both sides by $a + b$ (Theorem 2, Sec. 6.2), we have the desired result

$$a^3 + b^3 > a^2b + ab^2.$$

For some absolute inequalities, however, an analysis does not readily lead to a known relation. In such cases, we may have to experiment and try, at least tentatively, some known relations which may possibly lead to the desired result. This is illustrated in

Example 2. If a and b are unequal positive numbers, show that

$$a^2 + b^2 + 1 > ab + a + b.$$

SOLUTION. An analysis of the desired inequality does not suggest any known relations. However, the three expressions $(a - b)^2$, $(a - 1)^2$, and $(b - 1)^2$ involve all the terms in the inequality. Now, since $a \neq b$, $(a - b)^2$ is positive. Furthermore, while either a or b may be equal to 1, both cannot be equal to 1 at the same time, for $a \neq b$. Hence at least one of the expressions $(a - 1)^2$ and $(b - 1)^2$ must always be positive, and both are always non-negative. Thus, we are justified in taking the sum of these three expressions as positive, and we write

$$(a - b)^2 + (a - 1)^2 + (b - 1)^2 > 0,$$

with the expectation that this relation may lead to the desired result. Expanding, we have

$$a^2 - 2ab + b^2 + a^2 - 2a + 1 + b^2 - 2b + 1 > 0.$$

Collecting terms, $2a^2 + 2b^2 + 2 - 2ab - 2a - 2b > 0$.
Dividing by 2 (Theorem 2, Sec. 6.2), $a^2 + b^2 + 1 - ab - a - b > 0$.
Transposing (Cor. 1, Theorem 1, Sec. 6.2), $a^2 + b^2 + 1 > ab + a + b$, which is the desired result.

EXERCISES. GROUP 21

1. Prove that the sum of any positive number (except unity) and its reciprocal is greater than 2.

2. If a and b are unequal positive numbers, show that

$$\frac{a+b}{2} > \frac{2ab}{a+b}.$$

3. If a and b are positive and $a > b$, show that $\sqrt{a} > \sqrt{b}$ by a method independent of Corollary 3 of Theorem 6 (Sec. 6.2).

4. If a and b are unequal positive numbers, show that $a/b^2 + b/a^2 > 1/a + 1/b$.

5. If a and b are unequal positive numbers, show that $a + b < a^2/b + b^2/a$.

6. If a and b are unequal positive numbers, show that $a + b > 2\sqrt{ab}$.

7. If a and b are positive numbers and $a > b$, show that $\dfrac{a^2 - b^2}{a^2 + b^2} > \dfrac{a - b}{a + b}$.

8. If a, b, and c are all positive numbers, show that $(a + b + c)^2 > a^2 + b^2 + c^2$.

9. If a, b, and c are unequal positive numbers, show that $a^2 + b^2 + c^2 > ab + ac + bc$.

10. If a, b, and c are unequal positive numbers, show that $(a + b + c)^2 < 3(a^2 + b^2 + c^2)$.

11. If a, b, and c are unequal positive numbers, show that

$$(a + b)(b + c)(c + a) > 8abc.$$

12. If a and b are unequal positive numbers, show that $(a^3 + b^3)(a + b) > (a^2 + b^2)^2$.

13. If a and b are unequal numbers, show that $a^4 + b^4 > a^3b + ab^3$.

14. If a and b are unequal numbers, show that $(a^4 + b^4)(a^2 + b^2) > (a^3 + b^3)^2$.

15. If a, b, and c are unequal positive numbers, show that

$$ab(a + b) + bc(b + c) + ca(c + a) > 6abc.$$

16. If a and b are positive numbers and $a > b$, show that $a^4 - b^4 < 4a^4 - 4a^3b$.

17. Determine the values of a for which $a^3 + 1 > a^2 + a$.

18. If a and b are positive numbers, determine which is the greater, $\dfrac{a + 2b}{a + 3b}$ or $\dfrac{a + b}{a + 2b}$.

19. If a, b, c, and d are unequal positive numbers, and if $\dfrac{a}{b} > \dfrac{c}{d}$, show that $\dfrac{a}{b} > \dfrac{a + c}{b + d} > \dfrac{c}{d}$.

20. If a, b, x, and y are unequal positive numbers such that $a^2 + b^2 = 1$ and $x^2 + y^2 = 1$, show that $ax + by < 1$.

21. If a, b, c, x, y, and z are unequal positive numbers such that $a^2 + b^2 + c^2 = 1$ and $x^2 + y^2 + z^2 = 1$, show that $ax + by + cz < 1$.

22. If a, b, and c are unequal positive numbers, show that $2(a^3 + b^3 + c^3) > a^2b + b^2a + b^2c + c^2b + c^2a + a^2c$. *Hint:* Use the result of Example 1 (Sec. 6.3).

23. If a, b, and c are unequal positive numbers, show that $(a + b - c)^2 + (b + c - a)^2 + (c + a - b)^2 > ab + bc + ca$. *Hint:* Use the result of Ex. 9.

24. If a, b, c, x, y, and z are unequal positive numbers, show that

$$(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) > (ax + by + cz)^2.$$

25. If a, b, and c are unequal positive numbers, show that $a^3 + b^3 + c^3 > 3abc$. *Hint:* Use the result of Ex. 9.

6.4. CONDITIONAL LINEAR INEQUALITIES

In this chapter we will consider only conditional inequalities involving a single variable, say x. Our problem then is to determine the range of values of the variable x for which the inequality holds; this range is called the *solution* of the inequality. If the variable x occurs to only the first power, the inequality is said to be *linear*. The solution of a linear inequality is very simple and is analogous to the solution of the linear equation in one variable (Sec. 4.4).

Example. Solve the linear inequality $x + 1 > 3x + 5$, and illustrate the result graphically.

SOLUTION. We are to find the values of x for which

$$x + 1 > 3x + 5. \tag{1}$$

As in linear equations, we transpose all terms in x to one side and all known terms to the other side. We thus obtain

$$-2x > 4.$$

Dividing by -2, $\quad x < -2. \quad$ (Theorem 3, Sec. 6.2)

This is the required solution, which states that for all values of x less than -2, the inequality (1) holds.

For the graphical representation of this result, we transpose all terms of (1) to the left side, giving us the equivalent inequality

$$-2x - 4 > 0. \tag{2}$$

We have here our first example of the significance of Corollary 2 of Theorem 1 (Sec. 6.2). The inequality (2) now tells us that for all values of x less than -2, the linear function $-2x - 4$ is positive. The graph of this

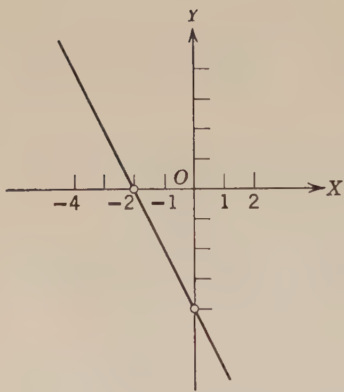


Figure 20

linear function is a straight line (Sec. 3.9) and is shown in Fig. 20. We also see here that the zero of the function is -2 and that for all values of $x < -2$, the line lies above the X-axis.

6.5. CONDITIONAL QUADRATIC INEQUALITIES

In Chapter 5 we considered the solution of the quadratic equation in one variable or the determination of the zeros of the quadratic function. By the solution of the conditional quadratic inequality in one variable, say x, we mean the determination of those values of x for which the inequality holds, that is, those values of x for which the quadratic function is *not* equal to zero but is either positive or negative as required by the inequality.

We have seen that, when possible, a quadratic equation is solved by factoring. Similarly, for a quadratic inequality, we factor the quadratic function, if possible, and determine its zeros which, although not solutions of the inequality, are nevertheless *critical values* of the solution, as we shall now explain.

Consider first the linear function in one variable, $x - r$, where r, a constant, is the zero of the function. If we assign x a value just slightly greater than r, the function is positive; if x is assigned a value slightly less than r, the function is negative. In other words, if the value of x deviates the slightest from above to below r, the sign of the function changes. For this reason r is appropriately called the *critical value* of the function $x - r$. Similarly, by obtaining the two linear factors of a quadratic function, we may obtain its two critical values.

The first step in the solution of a quadratic inequality is to transpose, if necessary, all terms to one side of the inequality giving us, say, the relation

$$(1) \qquad ax^2 + bx + c > 0.$$

The advantage in this step is that we are now not concerned with the magnitude of the left member of (1) but only with its sign (Cor. 2, Theorem 1, Sec. 6.2). Factoring this left member (relation (3), Sec. 5.5), we have

$$(2) \qquad a(x - r_1)(x - r_2) > 0,$$

where r_1 and r_2 are the critical values.

First, assume that x is greater than r_1, making the factor $x - r_1$ positive. If this same value of x also makes the other factor $x - r_2$ positive, their product (together with $a > 0$) is positive, the inequality (2) is satisfied; our assumption is correct; and $x > r_1$ is a solution of the inequality (1). If, however, this value of x makes $x - r_2$ negative, the product is negative; the inequality is not satisfied; our assumption is false; and a solution of the inequality is now $x < r_1$. It is easily verified that we obtain the same results by initially assuming that x is *smaller* than r_1. We go through a similar argument for the other critical value r_2. The resulting two inequalities constitute the solution of the inequality (1). We will illustrate this procedure by a concrete numerical example.

Example 1. Solve the inequality

$$3x^2 - 2x - 2 < 2x^2 - 3x + 4,$$

and illustrate the result graphically.

SOLUTION. We first transpose all terms to one side, say the left, and obtain the equivalent inequality

$$x^2 + x - 6 < 0.$$

Next, factoring the left member, we have

$$(x - 2)(x + 3) < 0,$$

and the critical values are 2 and -3.

First, assume that $x > 2$. Then for values of x slightly greater than 2, both factors are positive and their product is positive, a result contrary to the requirement of the inequality. Hence our assumption that $x > 2$ is false; the correct solution is $x < 2$. Note that if we had assumed initially that $x < 2$, then for values of x slightly less than 2, the first factor is negative; the second factor is positive; their product is negative; and the inequality is satisfied.

Similarly, assume that $x > -3$. Then for values of x slightly greater than -3, the first factor is negative; the second factor is positive; their product is negative; the inequality is satisfied; and the solution is $x > -3$.

Hence the complete solution is $x < 2$, $x > -3$, that is, the given inequality holds for all values of x less than 2 but greater than -3. This solution may be written in one statement as $2 > x > -3$, which means all values of x between -3 and 2. This range of values is conveniently shown graphically in Fig. 21 by the linear coordinate system (Sec. 3.7). The quadratic function $x^2 + x - 6$ is shown graphically in Fig. 22 in accordance with Sec. 5.8. This graph shows the zeros of the function as $x = 2, -3$; it also shows that the graph is above the X-axis for values

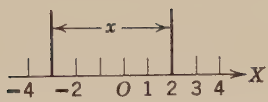


Figure 21

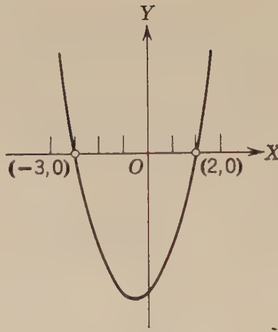


Figure 22

of $x > 2$ and < -3, and that the graph is below the X-axis for values of x between -3 and 2.

This method of solving an inequality by means of critical values may be used for any algebraic expression which may be factored into real linear factors. This is illustrated in

Example 2. Solve the inequality

$$(x + 1)(x - 2)(x - 3) > 0.$$

SOLUTION. The critical values are -1, 2, and 3. As in Example 1, we test each one by assuming values of x *slightly greater* than the critical value. We thus obtain the resulting signs of the factors, the sign of the product, and the resulting solution, as shown in the following table.

Assume	Signs of Factors	Product	Solution
$x > -1$	$+ \quad - \quad -$	>0	$x > -1$
$x > 2$	$+ \quad + \quad -$	<0	$x < 2$
$x > 3$	$+ \quad + \quad +$	>0	$x > 3$

The complete solution may therefore be written

$$-1 < x < 2, \qquad x > 3.$$

These ranges of values are shown graphically in Fig. 23.

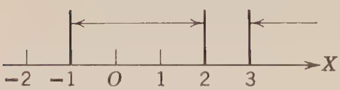


Figure 23

NOTE 1. The solution may also be conveniently effected by first plotting the critical values, as in Fig. 23, and then testing the given inequality for possible values of x in each of the four ranges or intervals shown: $x < -1$, $-1 < x < 2$, $2 < x < 3$, $x > 3$. The student should obtain the solution by this method. He should also solve Example 1 by this method.

We next consider the case of an inequality involving a quadratic function which cannot be factored into real linear factors. Although we ordinarily confine factoring to the field of rational numbers (Sec. 2.8), we do include irrational numbers for inequalities because they are real numbers. Thus, it is easy to see that the solution of the inequality $x^2 - 5 > 0$ is given by the relations $x > \sqrt{5}$, $x < -\sqrt{5}$. We will therefore now confine our attention to the quadratic function which is irreducible in the field of real numbers.

Let the quadratic function $ax^2 + bx + c$, $a \neq 0$ have a discriminant $b^2 - 4ac < 0$ so that the function is irreducible in the field of real numbers (Theorem 2, Sec. 5.5). By completing the square in x, we obtain (relation (1), Sec. 5.9)

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

or

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}. \tag{3}$$

In the right member of (3), $\left(x + \frac{b}{2a}\right)^2$ is non-negative for all values of x. Also, since $b^2 - 4ac < 0$, it follows that $4ac - b^2 > 0$, and hence $\frac{4ac - b^2}{4a}$ has the same sign as a. Accordingly, for all values of x, the right member of (3) is positive if $a > 0$ and is negative if $a < 0$. We record these results as

Theorem 7. *If the quadratic function*

$$ax^2 + bx + c, \qquad a \neq 0$$

has a negative discriminant $b^2 - 4ac$, then for all values of x the function is positive if $a > 0$ and is negative if $a < 0$.

NOTE 2. This theorem is very useful whenever one or more factors of an inequality are irreducible quadratic functions. Each such factor may be removed without any change except that, for a negative function, the sense must be reversed.

Example 3. Solve the inequalities

(a) $$x^2 + 2x + 5 > 0,$$

(b) $$2x - x^2 - 2 < 0,$$

and illustrate the results graphically.

SOLUTION. Since both functions have negative discriminants, it follows from Theorem 7 that with $a > 0$ in function (a), that function is positive for all values of x; and with $a < 0$ in function (b), that function is negative for all values of x. We can also see this by completing the square. Thus,

$$x^2 + 2x + 5 = (x + 1)^2 + 4 > 0 \text{ for all } x;$$

$$2x - x^2 - 2 = -(x^2 - 2x + 1) - 1 = -(x - 1)^2 - 1 < 0 \text{ for all } x.$$

Hence both inequalities are satisfied for all values of x. If the inequality signs were reversed, neither inequality would have a solution. The graphs of the two functions are shown in Fig. 24.

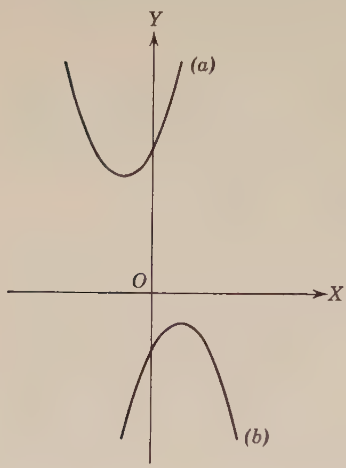


Figure 24

Finally, we consider a typê of conditional inequality which, although not a quadratic, may be solved by critical values.

Example 4. Solve the inequality

$$\frac{3}{x+2} > \frac{1}{x-1}.$$

SOLUTION. If the inequality sign were replaced by the equality sign, we would have a fractional equation whose solution would be effected by first multiplying both sides by the L.C.D. $(x + 2)(x - 1)$. The student may be tempted to do the same thing for this inequality, but if he does, he will run into difficulties. This is due to the fact that when we do not know the *sign* of a variable multiplier, we do not know whether we can retain the *sense* of the inequality or not (Theorem 3, Sec. 6.2). Hence we must never multiply or divide both sides of an inequality by a variable factor unless it retains the *same sign* throughout its entire range of values (Theorem 7).

Our first step, as usual, is to transpose all terms to one side of the inequality, whence

$$\frac{3}{x+2} - \frac{1}{x-1} > 0.$$

Our next step is *not* to multiply by the L.C.D. but to combine both fractions over their L.C.D. This gives us

$$\frac{2x - 5}{(x + 2)(x - 1)} > 0.$$

The critical values are $\frac{5}{2}$, -2, and 1. For each critical value, we assume x to be slightly greater or less than this value, and then observe the effects on the signs of the numerator and denominator and hence the sign of the fraction. The results are shown below in tabular form.

Assume	Signs of Numerator and Denominator	Sign of Fraction	Solution
$x > \frac{5}{2}$	$\frac{+}{+ \quad +}$	>0	$x > \frac{5}{2}$
$x > -2$	$\frac{-}{+ \quad -}$	>0	$x > -2$
$x > 1$	$\frac{-}{+ \quad +}$	<0	$x < 1$

The complete solution is therefore $x > \frac{5}{2}$, $-2 < x < 1$.

The student should show these ranges of values graphically. He should also solve the inequality by the method described in Note 1 of Example 2.

EXERCISES. GROUP 22

In each of Exs. 1–6, solve the linear inequality and illustrate the result graphically.

1. $x - 5 > 3 - x$.
2. $x + 4 < 3$.
3. $2x + 1 < 3x - 1$.
4. $4x + 10 > 4 - 2x$.
5. $x - \frac{2}{3} > 2x + \frac{4}{3}$.
6. $x + \frac{1}{2} < 2 + \frac{x}{4}$.

In each of Exs. 7–10, determine the values of x for which the given quadratic function is positive, negative, and zero. Illustrate the results graphically.

7. $2 + x - x^2$.
8. $x^2 - 6x + 8$.
9. $x^2 + 4x + 6$.
10. $4x - x^2 - 5$.

In each of Exs. 11–20, solve the given inequality and illustrate the result graphically.

11. $x^2 - x - 6 > 0$.
12. $x^2 + 5x + 4 < 0$.
13. $5x^2 + 8x - 3 > x^2 - 3x$.
14. $2x^2 + 5x - 1 < 2x + 1$.
15. $x^2 + 12x + 60 > 10 - 2x$.
16. $12x + x^2 - 30 > 2x^2 + 7$.
17. $2x^2 - 4x - 3 < 3x^2 + 2$.
18. $x^2 - 6x + 25 < 11$.
19. $x^2 + 8x - 11 < 2x^2 + 5$.
20. $x^2 - 8x + 8 > 4 - 4x$.

In each of Exs. 21–24. determine the values of x for which the given radical represents a real number.

21. $\sqrt{x - 7}$.
22. $\sqrt{x^2 + 16}$.
23. $\sqrt{x^2 - 16}$.
24. $\sqrt{x^2 + x - 12}$.

In each of Exs. 25–26, determine the values of k for which the roots of the given quadratic equation are real and unequal.

25. $4x^2 - kx + 1 = 0$.
26. $kx^2 + 2kx - 5 = 0$.

In each of Exs. 27–28, determine the values of k for which the roots of the given quadratic equation are complex.

27. $x^2 + kx - k = 0$.

28. $(k + 1)x^2 - 2kx + 1 = 0$.

In each of Exs. 29–30, determine the values of k for which the given system will have two distinct real solutions.

29. $x - 2y + k = 0,$
$x^2 + y^2 = 5.$

30. $x + 2y + k = 0,$
$y^2 - 2x + 6y + 9 = 0.$

In each of Exs. 31–50, solve the given inequality.

31. $x(x + 2)(x - 1) > 0$.

32. $(x + 1)(2x - 1)(x + 3) < 0$.

33. $x^3 - 2x^2 - x + 2 < 0$.

34. $x^3 + x^2 - 4x - 4 > 0$.

35. $(x + 2)(x - 1)(x - 4) < 0$.

36. $(x + 2)(x - 1)^2(x - 4) \gtrless 0$.

37. $(x^2 + x + 1)(x - 1) > 0$.

38. $(x^2 + 2x + 4)(x^2 - x - 2) < 0$.

39. $(x^2 + 2x - 3)(3x - 4 - x^2) > 0$.

40. $(x - 2 - x^2)(x^2 + 2x - 8) < 0$.

41. $\dfrac{x + 2}{x - 1} < 0$.

42. $\dfrac{x + 3}{x - 4} > 0$.

43. $\dfrac{x + 3}{x - 4} > 1$.

44. $\dfrac{2}{x - 2} < \dfrac{4}{x + 1}$.

45. $\dfrac{3}{2x - 2} > \dfrac{1}{2x + 1}$.

46. $\dfrac{x^2 - 4}{1 - x^2} > 2$.

47. $\dfrac{x}{x + 1} > \dfrac{x - 1}{x + 2}$.

48. $\dfrac{6}{x - 1} - \dfrac{5}{x - 2} > -2$.

49. $\dfrac{x^2 + x + 1}{x(x - 1)(x - 2)} > 0$.

50. $\dfrac{3}{x + 1} - \dfrac{5}{x - 1} > 6$.

6.6. OTHER CONDITIONAL INEQUALITIES

In this section we shall discuss several additional types of conditional inequalities. We will first consider an inequality which involves the absolute value (Sec. 2.4) of an expression, for example, the inequality $|x - 1| < 1$. Such inequalities arise in connection with the determination of what is known as the *interval of convergence* of a power series.

Example 1. Solve the inequality

$$|x - 1| < 1.$$

SOLUTION. The given inequality means precisely that

$$-1 < x - 1 < 1.$$

The solution of the inequality

$$-1 < x - 1$$

is readily found to be $x > 0$.

Similarly, the solution of the inequality

$$x - 1 < 1$$

is found to be $x < 2$.

Hence the solution of the given inequality is $0 < x < 2$.

In the next example we consider one of the fundamental properties of absolute values.

Example 2. If a and b are any real numbers, prove that

$$|a + b| \leq |a| + |b|.$$

SOLUTION. This inequality may, of course, be established by considering the various cases which arise: a and b both positive or both negative; a positive and b negative, and vice versa; and the various combinations where either a or b or both are zero. We will, however, give another proof.

Assume, contrary to the required inequality, that

$$|a + b| > |a| + |b|.$$

Squaring both sides (Cor. 2, Theorem 6, Sec. 6.2), we have

$$a^2 + 2ab + b^2 > a^2 + 2|a| \cdot |b| + b^2$$

whence

$$ab > |a| \cdot |b|,$$

which is not true for any values of a and b. This contradiction shows that our assumption is false, and the required inequality is established.

We next consider inequalities involving radicals. For some of these inequalities, care must be exercised in connection with signs. We must also remember that we are always dealing with real numbers.

Example 3. Solve the inequality

$$\sqrt{x - 1} + 2 > 0.$$

SOLUTION. In view of our experience with radical equations (Art. 46), we might isolate the radical and write

$$\sqrt{x - 1} > -2.$$

But we cannot square now, for both sides are not positive (Cor. 2, Theorem 6, Sec. 6.2).

If we square the original inequality as it stands, we have

$$x - 1 + 4\sqrt{x - 1} + 4 > 0$$

or

$$x + 3 + 4\sqrt{x - 1} > 0,$$

and we are in the same difficulty as before.

We now examine the original inequality more critically and observe that since the term 2 is already greater than zero, the only requirement for the radical $\sqrt{x-1}$ is that it represent a real non-negative number. This means that $x - 1 \geq 0$ or $x \geq 1$, which is therefore the solution.

EXERCISES. GROUP 23

In each of Exs. 1–8, solve the given inequality.

1. $|x| < 2$.
2. $|x| > 5$.
3. $|x - 2| < 1$.
4. $|x + 2| > 1$.
5. $|x - 5| < 1$.
6. $\left|\frac{x}{3}\right| < 1$.
7. $\left|\frac{x+2}{2}\right| < 1$.
8. $\left|\frac{x+1}{3}\right| < 1$.
9. By considering the various cases, establish the inequality of Example 2 (Sec. 6.6).
10. If a and b are any real numbers, prove that $|a - b| \leq |a| + |b|$.
11. If a and b are any real numbers, prove that $|a + b| \geq |a| - |b|$.
12. If a and b are any real numbers, prove that $|a - b| \geq |a| - |b|$.

In each of Exs. 13–22, solve the given inequality.

13. $\sqrt{x+1} > 2$.
14. $\sqrt{1-x} > 2$.
15. $\sqrt{x-1} < 1$.
16. $\sqrt{x-2} + 1 > 0$.
17. $\frac{1}{\sqrt{x+1}} < 2$.
18. $\frac{1}{\sqrt{x-1}} > 2$.
19. $\sqrt{x+5} + \sqrt{x} > 5$.
20. $\sqrt{x+4} - \sqrt{x-1} > 1$.
21. $\sqrt{3x+7} - \sqrt{x-2} > 3$.
22. $\sqrt{x+7} - \sqrt{x-1} > 2$.
23. If a and b are unequal positive numbers, show that $\frac{a+b}{2} > \sqrt{ab}$.
24. If a and b are unequal positive numbers, show that $\sqrt{ab} > \frac{2ab}{a+b}$.
25. If a and b are positive numbers, show that $\sqrt{a^2 + b^2} < a + b$.
26. If a and b are positive numbers, show that $\sqrt{a+b} < \sqrt{a} + \sqrt{b}$.

In each of Exs. 27–30, verify the given inequality without using a table of square roots.

27. $\sqrt{7} + \sqrt{3} > \sqrt{19}$.
28. $\sqrt{2} + \sqrt{6} < \sqrt{15}$.
29. $\sqrt{8} + \sqrt{6} < 2 + \sqrt{11}$.
30. $\sqrt{3} - \sqrt{7} > \sqrt{5} - \sqrt{10}$.

7

Mathematical induction. The binomial theorem

7.1. INTRODUCTION

As its title indicates, this chapter considers two distinct topics. The reason is that the *binomial theorem* is established by a method of proof known as *mathematical induction* or *complete induction*. The student must not infer from this that mathematical induction is merely a step in proving the binomial theorem. We shall see that there are a great variety of relations which may be proved by mathematical induction. In fact, we shall use mathematical induction in the next chapter to establish an important relation known as *De Moivre's Theorem*.

7.2. THE NATURE OF MATHEMATICAL INDUCTION

Instead of giving a formal definition of mathematical induction at the start, we shall discuss a very simple example to illustrate the logic underlying this method of proof.

Let us, therefore, consider the sum S_n of the first n odd tintegers,hat is,

$$S_n = 1 + 3 + 5 + \cdots + (2n - 1),$$

where $2n - 1$ represents the nth term in this sum. Initially, we write out the actual sum for the first four cases. Thus,

$$n = 1, \qquad S_1 = 1.$$
$$n = 2, \qquad S_2 = 1 + 3 = 4 = 2^2.$$
$$n = 3, \qquad S_3 = 1 + 3 + 5 = 9 = 3^2.$$
$$n = 4, \qquad S_4 = 1 + 3 + 5 + 7 = 16 = 4^2.$$

We have here indicated the sum for each case as the square of the number of terms in order to point up the obvious inference that the sum of n terms is *probably* equal to n^2. Note that we have not proved this relation for the sum of *any number* of terms; we have merely shown it to be true for values of n up to 4.

The method of mathematical induction now introduces the ingenious device of assuming that the relation is true for some one value of n, say k, and then attempting to show that, on the basis of this assumption, the relation is also true for $k + 1$, the next higher value of n. If this step is accomplished, the argument for completing the proof proceeds as follows. Since the relation has been shown to be true for $n = 1$, it follows from the preceding step that it is true for $n = 2$. Similarly, if it is true for $n = 2$, it is true for $n = 3$, and so on for every positive integral value of n.

Let us now complete the proof, by mathematical induction, of the relation

$$(1) \qquad 1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

Assume that (1) is true for $n = k$, that is,

$$(2) \qquad 1 + 3 + 5 + \cdots + (2k - 1) = k^2$$

is an assumed equality.

We now add the $(k + 1)$th term, $2(k + 1) - 1 = 2k + 1$, to both sides of (2). We thus have the equality

$$(3) \quad 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = k^2 + 2k + 1 = (k + 1)^2.$$

Now (3), *which is true if and only if* (2) *is true*, is a verification of relation (1) for $n = k + 1$. We have thus proved that if relation (1) is true for $n = k$, it is true for $n = k + 1$. Then the argument proceeds as above. Since relation (1) has been shown to be true for $n = 1$, it follows from (2) and (3) that it is true for $n = 2$. Similarly, if (1) is true for $n = 2$, it is true for $n = 3$, and so on for all positive integral values of n.

For convenience and future reference we now give a formal statement of

Mathematical Induction

Mathematical induction or complete induction is a form of reasoning which may be used to prove relations or statements depending upon some variable, say n, which assumes only positive integral values. The method of mathematical induction for proving a particular relation consists essentially of the following three steps:

1. The relation must be verified for $n = 1$ or the first value of n for which the relation is to hold.

2. Based upon the assumption that the relation is true for some value of n, say k, the relation must be shown to be true for $n = k + 1$.

3. The relation having been shown to be true for $n = 1$ in step 1, it follows from step 2 that it is true for $n = 2$. Similarly, if the relation is true for $n = 2$, it is true for $n = 3$, and so on for all positive integral values of n.

It must be emphasized that *both* steps 1 and 2 are essential for a valid proof. Step 3, of course, is a logical consequence of steps 1 and 2.

NOTE. We give here two illustrations of how a relation is shown to be invalid because it does not satisfy *both* steps 1 and 2.

Consider first the relation

$$1 + 3 + 5 + \cdots + (2n - 1) = n.$$

This relation is obviously true for $n = 1$ and hence satisfies step 1. But it is easy to show that it does not satisfy step 2. Accordingly, relation (4) does not hold for all positive integral values of n.

Consider next the relation

$$(5) \qquad 1 + 3 + 5 + \cdots + (2n - 1) = n^2 + 1.$$

It may be easily shown that this relation satisfies step 2. But it does not satisfy step 1 and hence does not hold for any positive integral value of n.

7.3. EXAMPLES OF MATHEMATICAL INDUCTION

In this article we give several illustrative examples and a comprehensive list of exercises in mathematical induction.

Example 1. By the method of mathematical induction, prove the relation

$$(1) \qquad 1^2 + 2^2 + 3^2 + \cdots + n^2 = \tfrac{1}{6}n(n + 1)(2n + 1),$$

where n is any positive integer.

SOLUTION. We will carry out, in order, each of the three steps outlined in Sec. 7.2.

1. Substitute $n = 1$ in relation (1). We obtain

$$1^2 = \frac{1}{6} \cdot 1(1 + 1)(2 + 1) = \frac{1}{6} \cdot 2 \cdot 3 = 1.$$

Hence step 1 is satisfied.

2. Assume (1) is true for $n = k$, that is, assume the following relation is true:

$$(2) \qquad 1^2 + 2^2 + 3^2 + \cdots + k^2 = \tfrac{1}{6}k(k + 1)(2k + 1).$$

Add the $(k + 1)$th term, $(k + 1)^2$, to both sides of (2). This gives us the equality

$$(3)\quad 1^2 + 2^2 + 3^2 + \cdots + k^2 + (k + 1)^2 = \tfrac{1}{6}k(k + 1)(2k + 1) + (k + 1)^2.$$

We must now show that the right member of (3) is identical with the right member of (1) when n is replaced by $k + 1$. Thus, first factoring out $\dfrac{k+1}{6}$, we have

$$\begin{aligned}
&\tfrac{1}{6}k(k + 1)(2k + 1) + (k + 1)^2\\
&\quad = \frac{k+1}{6}[k(2k + 1) + 6(k + 1)]\\
&\quad = \frac{k+1}{6}[2k^2 + 7k + 6] = \frac{k+1}{6}(k + 2)(2k + 3)\\
&\quad = \frac{k+1}{6}[(k + 1) + 1][2(k + 1) + 1].
\end{aligned}$$

This last expression is identical with the right member of (1) when n is replaced by $k + 1$. We have therefore proved that if (1) is true for $n = k$, it is also true for $n = k + 1$, and step 2 is satisfied.

3. Since (1) is true for $n = 1$ by step 1, it follows from step 2 that (1) is true for $n = 2$. For the same reason, if it is true for $n = 2$, it is true for $n = 3$, and so on for all positive integral values of n.

This completes the proof.

Example 2. By the method of mathematical induction, prove that $x^{2n} - y^{2n}$ is exactly divisible by $x + y$ for every positive integral value of n.

SOLUTION. 1. For $n = 1$, we have $(x^2 - y^2)/(x + y) = x - y$, and step 1 is satisfied.

2. We are now to prove that if $x^{2k} - y^{2k}$ is exactly divisible by $x + y$, then $x^{2k+2} - y^{2k+2}$ is also exactly divisible by $x + y$. There are several ways of doing this. We will use the very natural method of actually dividing $x^{2k+2} - y^{2k+2}$ by $x + y$. By ordinary algebraic division (Sec. 2.7), we have

$$\frac{x^{2k+2} - y^{2k+2}}{x + y} = x^{2k+1} - x^{2k}y + \frac{x^{2k}y^2 - y^{2k+2}}{x + y},$$

where the division has been carried out just far enough to permit the use of the assumption that $x^{2k} - y^{2k}$ is exactly divisible by $x + y$. Thus, for the remainder we have

$$x^{2k}y^2 - y^{2k+2} = y^2(x^{2k} - y^{2k}),$$

so that, in view of our assumption, the remainder is exactly divisible by $x + y$, and hence the entire division of $x^{2k+2} - y^{2k+2}$ by $x + y$ is exact. Therefore, step 2 is satisfied.

3. Here we repeat the usual verbal statements to complete the proof for all positive integral values of n.

NOTE. Many of the difficulties encountered in step 2 of a proof by mathematical induction will be avoided if the result of substituting $k + 1$ for n is so manipulated as to permit the use of the assumption for $n = k$. The student should observe this fact in the preceding example.

EXERCISES. GROUP 24

1. Show that relation (4) of Sec. 7.2 does not satisfy step 2 of the method of mathematical induction.

2. Show that relation (5) of Sec. 7.2 satisfies step 2 of the method of mathematical induction.

3. In Example 2 of Sec. 7.3, carry out step 2 by using the identity
$$x^{2k+2} - y^{2k+2} \equiv x^2(x^{2k} - y^{2k}) + y^{2k}(x^2 - y^2).$$

4. Prove that $x^{2n-1} + y^{2n-1}$ is exactly divisible by $x + y$ for every positive integral value of n, carrying out step 2 as in Example 2 of Sec. 7.3.

5. Carry out step 2 of Ex. 4 by using the identity
$$x^{2k+1} + y^{2k+1} \equiv x^2(x^{2k-1} + y^{2k-1}) - y^{2k-1}(x^2 - y^2).$$

6. Prove that $x^n - y^n$ is exactly divisible by $x - y$ for every positive integral value of n, carrying out step 2 as in Example 2 of Sec. 7.3.

7. Carry out step 2 of Ex. 6 by using the identity
$$x^{k+1} - y^{k+1} \equiv x(x^k - y^k) + y^k(x - y).$$

In each of Exs. 8–39, by the method of mathematical induction, prove the given relation or statement, n being a positive integer.

8. $1 + 2 + 3 + \cdots + n = \dfrac{n(n + 1)}{2}$.

9. $2 + 4 + 6 + \cdots + 2n = n(n + 1)$.

10. $1 + 4 + 7 + \cdots + (3n - 2) = \dfrac{n}{2}(3n - 1)$.

11. $3 + 6 + 9 + \cdots + 3n = \frac{3}{2}n(n + 1)$.

12. $5 + 10 + 15 + \cdots + 5n = \frac{5}{2}n(n + 1)$.

13. $a + (a + d) + (a + 2d) + \cdots + [a + (n - 1)d] = \dfrac{n}{2}[2a + (n - 1)d]$.

14. $2 + 2^2 + 2^3 + \cdots + 2^n = 2(2^n - 1)$.

15. $3 + 3^2 + 3^3 + \cdots + 3^n = \frac{3}{2}(3^n - 1)$.

16. $1 + 5 + 5^2 + \cdots + 5^{n-1} = \frac{1}{4}(5^n - 1)$.

17. $1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} = 2 - \frac{1}{2^{n-1}}.$

18. $a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1 - r^n)}{1 - r}.$

19. $1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2 = \frac{n}{3}(4n^2 - 1).$

20. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2}{4}(n + 1)^2.$

21. $(1 + 2 + 3 + \cdots + n)^2 = \frac{n^2}{4}(n + 1)^2.$

22. $1^3 + 3^3 + 5^3 + \cdots + (2n - 1)^3 = n^2(2n^2 - 1).$

23. $1 + 3 + 6 + \cdots + \frac{n}{2}(n + 1) = \frac{n}{6}(n + 1)(n + 2).$

24. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n + 1) = \frac{n}{3}(n + 1)(n + 2).$

25. $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n + 2) = \frac{n}{6}(n + 1)(2n + 7).$

26. $2 \cdot 5 + 3 \cdot 6 + 4 \cdot 7 + \cdots + (n + 1)(n + 4) = \frac{n}{3}(n + 4)(n + 5).$

27. $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \cdots + n(n + 1)(n + 2)$
$= \frac{n}{4}(n + 1)(n + 2)(n + 3).$

28. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n + 1)} = \frac{n}{n + 1}.$

29. $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2n - 1)(2n + 1)} = \frac{n}{2n + 1}.$

30. $\frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{n(n + 2)} = \frac{n(3n + 5)}{4(n + 1)(n + 2)}.$

31. $1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + n \cdot 2^n = (n - 1)2^{n+1} + 2.$

32. $1 \cdot 1 + 2 \cdot 3^2 + 3 \cdot 5^2 + \cdots + n(2n - 1)^2 = \frac{n}{6}(n + 1)(6n^2 - 2n - 1).$

33. $1 \cdot 3 + 3 \cdot 3^2 + 5 \cdot 3^3 + \cdots + (2n - 1)3^n = (n - 1)3^{n+1} + 3.$

34. If a and b are positive numbers such that $a > b$, then $a^n > b^n$.

35. $2^{4n} - 1$ is divisible by 15.

36. $2^{2n} + 5$ is divisible by 3.

37. $3^{2n} + 7$ is divisible by 8.

38. $\frac{x^n - y^n}{x - y} = x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}.$

39. $\dfrac{x^n + y^n}{x + y} = x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \cdots - xy^{n-2} + y^{n-1}$, n odd.

40. Establish Theorem 12 of Sec. 2.6 by the method of mathematical induction.

7.4. THE BINOMIAL THEOREM

The binomial theorem is essentially a formula whereby we may write out each term in the expansion of a binomial raised to any positive integral power. To obtain some idea as to the formation of such an expansion for $(a + b)^n$, where n is a positive integer, we write out the expansion for the first four values of n. Thus, by actual multiplication, we have

$$(a + b)^1 = a + b,$$
$$(a + b)^2 = a^2 + 2ab + b^2,$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$
$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

We now observe that each expansion has the following characteristics:

1. The number of terms is $n + 1$, one more than the exponent n of the binomial.
2. The exponent of a in the first term is n and decreases by 1 in each succeeding term.
3. The quantity b appears initially in the second term with the exponent 1, and this exponent increases by 1 in each succeeding term. This exponent is always 1 less than the number of the term.
4. The sum of the exponents of a and b in any term is always equal to n.
5. The coefficients of a and b exhibit a certain type of symmetry, namely, the coefficients of terms equidistant from the ends are the same.
6. The coefficient of the first term is unity and that of the second term is n.
7. If, in any term, the coefficient is multiplied by the exponent of a and this product is divided by the exponent of b increased by 1, the result is the coefficient of the next term.

NOTE 1. The student will readily see the first six characteristics, but the seventh may not be so obvious. Since this is very important in determining the coefficients, we illustrate the process for the expansion of $(a + b)^4$. Thus the coefficient of the third term is obtained from the second term as follows. The coefficient 4 of the second term is multiplied by the exponent 3 of a and this product is divided by the exponent 1 of b increased by 1. That is, $\dfrac{4 \times 3}{1 + 1} = 6$, the coefficient of the third term. Similarly, from this coefficient we have $\frac{6 \times 2}{2 + 1} = 4$, the coefficient of the fourth term, and so on.

Before we attempt to write out the form of the general expansion of $(a + b)^n$, it will be convenient to have the following

Definition. By the symbol $n!$, called *factorial n*, we mean the product of all the consecutive positive integers from 1 to n. That is,

$$(1) \qquad n! = 1 \cdot 2 \cdot 3 \cdots n.$$

Thus, $$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24.$$

For completeness, it is often useful to have a value for $0!$ which is not defined by relation (1) where n is a positive integer. Hence we must have a separate definition for $0!$ which is motivated as follows. From (1) we have

$$(2) \qquad n! = n(n - 1)!$$

From relation (2), for $n = 1$, we have

$$1! = 1(0)!$$

Hence, for this relation to hold, we make the special definition

$$0! = 1.$$

NOTE 2. Factorial n is also frequently denoted by the symbol $\underline{n|}$. We shall, however, use the symbol $n!$.

If we now *assume* that, for *any* positive integral value of n, the expansion of $(a + b)^n$ has the same characteristics that we observed for $n = 1, 2, 3, 4$, then we may write

$$(a + b)^n = a^n + \frac{n}{1} a^{n-1}b + \frac{n(n-1)}{1 \cdot 2} a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3}b^3 + \cdots + \frac{n(n-1)\cdots(n-r+2)}{1 \cdot 2 \cdot 3 \cdots (r-1)} a^{n-r+1}b^{r-1} + \cdots + b^n,$$

which, in view of the definition of $n!$, may be written

$$(3) \qquad (a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!} a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!} a^{n-3}b^3 + \cdots + \frac{n(n-1)\cdots(n-r+2)}{(r-1)!} a^{n-r+1}b^{r-1} + \cdots + b^n,$$

where the rth term, $\frac{n(n-1)\cdots(n-r+2)}{(r-1)!}a^{n-r+1}b^{r-1}$, is also known as the *general term.*

The relation (3) is called the *binomial theorem* for positive integral exponents. It has been verified for $n = 1, 2, 3, 4$. The question now arises. Does this relation hold for all positive integral values of n? The answer is "yes" and is proved by mathematical induction as shown in the next article.

7.5. PROOF OF THE BINOMIAL THEOREM

For convenience, we rewrite the binomial theorem, including the $(r-1)$th term as well as the rth term. Thus,

$$(1) \qquad (a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots$$
$$+ \frac{n(n-1)\cdots(n-r+3)}{(r-2)!}a^{n-r+2}b^{r-2}$$
$$+ \frac{n(n-1)\cdots(n-r+2)}{(r-1)!}a^{n-r+1}b^{r-1}$$
$$+ \cdots + nab^{n-1} + b^n.$$

We will establish the validity of relation (1) for all positive integral values of n by the method of mathematical induction. In the previous section we verified (1) for $n = 1$ and have thus completed step 1 of the proof (Sec. 7.2).

In accordance with step 2 we assume that (1) holds for $n = k$, giving us the assumed equality

$$(2) \qquad (a+b)^k = a^k + ka^{k-1}b + \cdots$$
$$+ \frac{k(k-1)\cdots(k-r+3)}{(r-2)!}a^{k-r+2}b^{r-2}$$
$$+ \frac{k(k-1)\cdots(k-r+2)}{(r-1)!}a^{k-r+1}b^{r-1}$$
$$+ \cdots + kab^{k-1} + b^k.$$

We now multiply both sides of (2) by $a + b$. We obtain $(a+b)^{k+1}$ on the left side. In the next two lines we write, in order, the product of the right side of (2) by a and then by b.

(3)

$$a^{k+1} + ka^kb + \cdots + \frac{k(k-1)\cdots(k-r+2)}{(r-1)!}a^{k-r+2}b^{r-1} + \cdots + ab^k,$$

(4)

$$a^kb + \cdots + \frac{k(k-1)\cdots(k-r+3)}{(r-2)!}a^{k-r+2}b^{r-1} + \cdots + kab^k + b^{k+1}.$$

Adding (3) and (4), we have for the right member of $(a+b)^{k+1}$,

$$a^{k+1} + (k+1)a^kb$$
$$+ \cdots + \left[\frac{k(k-1)\cdots(k-r+2)}{(r-1)!} + \frac{k(k-1)\cdots(k-r+3)}{(r-2)!}\right]a^{k-r+2}b^{r-1}$$
$$+ \cdots + (k+1)ab^k + b^{k+1}.$$

The coefficient of the rth term in this last expansion may be simplified as follows:

$$\frac{k(k-1)\cdots(k-r+2)}{(r-1)!} + \frac{k(k-1)\cdots(k-r+3)}{(r-2)!}$$
$$= \frac{k(k-1)\cdots(k-r+3)}{(r-1)!}(k-r+2) + \frac{k(k-1)\cdots(k-r+3)}{(r-1)(r-2)!}(r-1)$$
$$= \frac{k(k-1)\cdots(k-r+3)}{(r-1)!}[k-r+2+r-1]$$
$$= \frac{k(k-1)\cdots(k-r+3)}{(r-1)!}[k+1] = \frac{(k+1)k(k-1)\cdots(k-r+3)}{(r-1)!}.$$

Hence we may finally write

(5) $$(a+b)^{k+1} = a^{k+1} + (k+1)a^kb$$
$$+ \cdots + \frac{(k+1)k\cdots(k-r+3)}{(r-1)!}a^{k-r+2}b^{r-1}$$
$$+ \cdots + (k+1)ab^k + b^{k+1}.$$

Comparing (1) and (5), and particularly the rth terms, we see that (5) is precisely the result obtained by replacing n by $k+1$ in (1). Hence we have shown that if the binomial theorem (1) is true for $n = k$, it is also true for $n = k + 1$. We have thus completed step 2.

Then, by the usual argument of step 3, it follows from the above results that the binomial theorem (1) holds for all positive integral values of n.

This completes the proof.

NOTE 1. It must be emphasized that we have proved the binomial theorem only for positive integral values of the exponent. It is shown by advanced methods that the binomial expansion of $(a + b)^n$ holds also for fractional and negative values of n provided that the absolute value of b/a is less than unity. Under these conditions the number of terms is unlimited, that is, the expansion continues indefinitely and we have what is termed an *infinite series*.

NOTE 2. In the fifth characteristic of the binomial expansion (Sec. 7.4), we noted a certain type of symmetry in the coefficients of the terms. This symmetry is clearly shown in the following triangular array known as *Pascal's triangle*, which gives the coefficients of the terms in the expansion of $(a + b)^n$ for values of n as indicated. These coefficients are known as the *binomial coefficients*.

$$\begin{array}{lccccccccccc}
n = 0 & & & & & & 1 & & & & & \\
n = 1 & & & & & 1 & & 1 & & & & \\
n = 2 & & & & 1 & & 2 & & 1 & & & \\
n = 3 & & & 1 & & 3 & & 3 & & 1 & & \\
n = 4 & & 1 & & 4 & & 6 & & 4 & & 1 & \\
n = 5 & 1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

In Pascal's triangle we observe that the end elements are all unity, for the binomial coefficients of the first and last terms are always unity. Each interior element may be obtained as the sum of the two elements in the row immediately above and to the left and right of that element. Thus for $n = 4$, the second binomial coefficient 4 is the sum of the elements 1 and 3 of the preceding row and, respectively, to the left and right of 4; similarly, the third binomial coefficient 6 is obtained as the sum of the elements 3 and 3 of the preceding row, and so on. This relation among the binomial coefficients will be proved in a later chapter on the subject of *permutations and combinations*.

We will now illustrate the binomial theorem by several examples.

Example 1. Expand $(a + 2b)^5$ by the binomial theorem.

SOLUTION. We start by writing the first term a^5 and the coefficient 5 of the second term, which is $a^4(2b)$. From this point on we can readily write out all the succeeding terms together with their binomial coefficients in accordance with the characteristics of the binomial expansion as noted in Sec. 7.4. Thus,

$$\begin{aligned}
(a + 2b)^5 &= a^5 + 5a^4(2b) + \frac{5 \cdot 4}{2} a^3(2b)^2 \\
&\quad + \frac{10 \cdot 3}{3} a^2(2b)^3 + \frac{10 \cdot 2}{4} a(2b)^4 + \frac{5 \cdot 1}{5} (2b)^5 \\
&= a^5 + 5a^4(2b) + 10a^3(2b)^2 + 10a^2(2b)^3 + 5a(2b)^4 + (2b)^5.
\end{aligned}$$

Note that we have always kept the term $2b$ in parentheses so as not to interfere with the correct formation of the binomial coefficients. Now we can expand the powers of $2b$ and obtain the final form:

$$(a + 2b)^5 = a^5 + 10a^4b + 40a^3b^2 + 80a^2b^3 + 80ab^4 + 32b^5.$$

Example 2. Expand $\left(\frac{2a}{x^2} - \frac{x^2}{2a}\right)^4$ by the binomial theorem.

SOLUTION. In this expansion it is desirable to enclose *both* terms in parentheses since we are now concerned not only with the correct formation of the binomial coefficients but also with the final exponents and sign of the individual terms. We therefore write out the expansion in steps as follows:

$$\begin{aligned}\left(\frac{2a}{x^2} - \frac{x^2}{2a}\right)^4 &= \left(\frac{2a}{x^2}\right)^4 + 4\left(\frac{2a}{x^2}\right)^3\left(-\frac{x^2}{2a}\right) + \frac{4\cdot 3}{2}\left(\frac{2a}{x^2}\right)^2\left(-\frac{x^2}{2a}\right)^2 \\ &\quad + \frac{6\cdot 2}{3}\left(\frac{2a}{x^2}\right)\left(-\frac{x^2}{2a}\right)^3 + \frac{4\cdot 1}{4}\left(-\frac{x^2}{2a}\right)^4 \\ &= \frac{16a^4}{x^8} - 4\frac{8a^3}{x^6}\cdot\frac{x^2}{2a} + 6\frac{4a^2}{x^4}\cdot\frac{x^4}{4a^2} - 4\frac{2a}{x^2}\cdot\frac{x^6}{8a^3} + \frac{x^8}{16a^4} \\ &= \frac{16a^4}{x^8} - \frac{16a^2}{x^4} + 6 - \frac{x^4}{a^2} + \frac{x^8}{16a^4}.\end{aligned}$$

7.6. THE GENERAL TERM

We have already noted (Sec. 7.4) that in the binomial expansion of $(a + b)^n$,

(1) $$\text{the } r\text{th term} = \frac{n(n-1)\cdots(n-r+2)}{(r-1)!}a^{n-r+1}b^{r-1},$$

and is called the *general term*. It is a convenient formula for finding any term in a binomial expansion without finding the preceding terms. It is worth noting that the coefficient in (1) has the same number of factors in both numerator and denominator, namely, $(r - 1)!$ factors.

It follows from (1) that the term containing b^r is the $(r + 1)$th term and that

(2) $$\text{the } (r+1)\text{th term} = \frac{n(n-1)\cdots(n-r+1)}{r!}a^{n-r}b^r,$$

which is often called the general term rather than the form (1). While either of these forms may be used to obtain a particular term in a binomial expansion, we will use form (1) at present. Later we will have occasion to use form (2) when we consider the binomial coefficients in terms of combinations.

Example 1. Find the fourth term in the expansion of $(a + 2b)^5$.

SOLUTION. Using form (1) above, we have

$$\text{4th term of } (a + 2b)^5 = \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} a^2(2b)^3 = 10a^2(8b^3) = 80a^2b^3.$$

(See Example 1 of Sec. 7.5.)

Example 2. In the expansion of $\left(2x^2 - \dfrac{xy}{2}\right)^9$, find the term involving x^{14}.

SOLUTION. This problem differs from the preceding one in that we do not know the number of the required term. Hence, let r represent the number of the term. In accordance with form (1), the rth term, aside from its coefficient, will involve $(2x^2)^{9-(r-1)}\left(-\dfrac{xy}{2}\right)^{r-1}$ so that the exponent of x in this term is $2(10 - r) + r - 1 = 20 - 2r + r - 1 = -r + 19$. Since we are concerned with an exponent of x equal to 14, we must have $-r + 19 = 14$, whence $r = 5$. That is, the required term is the

$$\text{5th term of } \left(2x^2 - \frac{xy}{2}\right)^9 = \frac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4}(2x^2)^5\left(-\frac{xy}{2}\right)^4 = 126(32x^{10})\left(\frac{x^4y^4}{16}\right)$$
$$= 252x^{14}y^4.$$

EXERCISES. GROUP 25

In each of Exs. 1–14, perform the binomial expansion indicated.

1. $(3a - b)^4$.
2. $(x - 2y)^5$.
3. $(x + 3y)^6$.
4. $(x^2 - y^2)^4$.
5. $(x^2 + x^{1/2})^4$.
6. $(x^2 - x^{-2})^5$.
7. $\left(\dfrac{a}{2} - \dfrac{2}{a}\right)^4$.
8. $\left(\dfrac{a^2}{2} + \dfrac{2}{a^2}\right)^6$.
9. $\left(\dfrac{x^{1/2}}{y} - \dfrac{y}{x^{1/2}}\right)^4$.
10. $(a\sqrt{b} + b\sqrt{a})^6$.
11. $(a^{3/2} - a^{-3/2})^4$.
12. $(\sqrt{3} - \sqrt{2})^4$.
13. $(a + b - c)^3$.
14. $(1 + x)^4 + (1 - x)^4$.

In each of Exs. 15–26, write and simplify the first four terms of the binomial expansion.

15. $(2a - b)^7$.
16. $\left(x + \dfrac{y}{2}\right)^8$.
17. $\left(a - \dfrac{b}{3}\right)^9$.
18. $\left(a + \dfrac{x}{2}\right)^{10}$.
19. $(x^{1/2} - y^{1/2})^{12}$.
20. $(1 + x)^{20}$.
21. $(1 + x)^{-1}$.
22. $(1 + x^2)^{-1}$.
23. $(1 - x)^{-2}$.
24. $(1 + x)^{1/2}$.
25. $(1 - x^2)^{1/2}$.
26. $(1 + x)^{3/2}$.

27. Obtain the result of Ex. 21 by dividing 1 by $1 + x$.
28. Obtain the result of Ex. 22 by dividing 1 by $1 + x^2$.
29. Evaluate $(1.01)^4$ by expanding $(1 + 0.01)^4$.
30. Evaluate $(0.99)^3$ by using a binomial expansion.
31. Evaluate $\sqrt{0.99}$ correctly to 3 decimal places by using the result of Ex. 25
32. Continue Pascal's triangle (Sec. 7.5) for $n = 6, 7, 8$.
33. Show that the coefficient of the rth term of $(a + b)^n$, as given by relation (1) of Sec. 7.6, may be written in the form $\dfrac{n!}{(n - r + 1)!\,(r - 1)!}$.

34. Show that the coefficient of the rth term of $(a + b)^n$, as given by relation (1) of Sec. 7.6, holds for all values of r except 1, but that the form given in Ex. 33 does hold for $r = 1$.

35. Show that the coefficient of the $(r + 1)$th term of $(a + b)^n$, as given by relation (2) of Sec. 7.6, may be written in the form $\dfrac{n!}{r!\,(n - r)!}$.

36. Show that the sum of the coefficients in the expansion of $(a + b)^n$ is equal to 2^n.

37. Verify the property of the coefficients in the binomial expansion, as given by Ex. 36, in Pascal's triangle.

In each of Exs. 38–49, find only the specified term (s) in the given binomial expansion.

38. Fourth term of $(a - 2b)^9$.
39. Eighth term of $(x^{1/2} + y^{1/3})^{12}$.
40. Fifth term of $\left(x + \dfrac{y}{2}\right)^7$.
41. Seventh term of $\left(\dfrac{a}{2} - x\right)^{11}$.
42. Middle term of $\left(\dfrac{x}{y} + \dfrac{y}{x}\right)^8$.
43. Middle term of $\left(\dfrac{a}{b} - \dfrac{b}{a}\right)^{10}$.
44. Two middle terms of $\left(\dfrac{x^2}{2} - y\right)^9$.
45. Two middle terms of $(ab + \frac{1}{2})^{11}$.
46. Term involving a^7 in $\left(\dfrac{a}{3} + 9b\right)^{10}$.
47. Term involving y^4 in $\left(\dfrac{2x}{3y} + \dfrac{3y}{2x}\right)^{10}$.
48. Term free of x in $\left(\dfrac{2x}{3} - \dfrac{3}{2x}\right)^6$.
49. Term free of x in $\left(\dfrac{x^{1/2}}{y^{2/3}} + \dfrac{y^{1/3}}{x^{3/2}}\right)^{16}$.
50. Show that the middle term in the expansion of $(1 + x)^{2n}$ may be written in the form $\dfrac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{n!} 2^n x^n$.

8

Complex numbers

8.1. INTRODUCTION

Up to this point our work, with few exceptions, has been confined to the real number system. We have, however, previously noted the need for complex numbers. In fact, in our first discussion of the numbers of algebra (Sec. 1.3), we arrived at the conclusion that the system of complex numbers was to be considered the number system of algebra. It is therefore the object of this chapter to present a formal study of complex numbers and their properties.

Although sufficient material has already been developed to perform many of the operations with complex numbers, we will find it extremely useful and convenient to introduce and use a trigonometric form of the complex number. This will require some knowledge of plane trigonometry on the part of the student. In Appendix I we have therefore included the necessary definitions and formulas of trigonometry.

In previous chapters we have given some definitions and comments pertaining to complex numbers. For convenience and completeness, several of these statements will be repeated and included in the next section.

8.2. DEFINITIONS AND PROPERTIES

In the solution of the simple quadratic equation $x^2 + 1 = 0$, we seek a number x which meets the condition that $x^2 = -1$, a negative number. But from the rule of signs for the multiplication of real numbers (Sec. 2.5), it follows that every real number has the property that its square is a non-negative real number. Hence the number x sought as a solution of $x^2 + 1 = 0$ cannot be a real number. To meet this situation and to make the solution of the equation possible, we introduce a new number for which we have the

Definition. The quantity $\sqrt{-1}$ is called the *imaginary unit*, is represented by the symbol i, and has the property that $i^2 = -1$.

To represent the square root of negative numbers other than -1, we introduce a new class of numbers for which we have the

Definition. A number of the form bi, where b is any real number and i is the imaginary unit, is called a *pure imaginary number*.

In connection with our study of the quadratic equation (Sec. 5.5), we saw that under certain conditions the roots of such an equation are numbers expressed as the sum of a real number and a pure imaginary number. Accordingly we have the

Definition. A number of the form $a + bi$, where a and b are real numbers and i is the imaginary unit, is called a *complex number*.

If $a = 0$ but $b \neq 0$, the complex number $a + bi$ assumes the form bi so that the pure imaginary number is a special case of the complex number.

If $b = 0$, the complex number $a + bi$ assumes the form a, a real number. In view of this fact it will be recalled that at the end of Sec. 1.4 we stated that a real number is merely a special case of a complex number; consequently, the set of all real numbers is said to be a *subclass* of the system of complex numbers.

Definition. Two complex numbers $a + bi$ and $c + di$ are said to be *equal* if and only if $a = c$ and $b = d$.

As an immediate consequence of this definition, it follows that $a + bi = 0$ if and only if $a = 0$ and $b = 0$.

An illustration of the use of this definition is given by the following

Example. Find real values of x and y such that the following relation will hold

$$x^2 + 2y^2 + xi + yi = xy + 7 + 3i.$$

SOLUTION. We first rearrange the terms so that each side will be in the form of the complex number $a + bi$. Thus,

$$(x^2 + 2y^2) + (x + y)i = (xy + 7) + 3i.$$

Now, by the definition of the equality of two complex numbers, we have for the real and the imaginary parts, respectively,

$$x^2 + 2y^2 = xy + 7,$$

$$x + y = 3.$$

By the method of Sec. 5.12, the solutions of this system are readily found to be $x = 1$, $y = 2$ and $x = \frac{11}{4}$, $y = \frac{1}{4}$, and these are the required values.

We have previously noted (Sec. 6.1) that the order relation of real numbers does not apply to complex numbers, that is, we do not speak of one complex number being greater or less than another. Consequently, complex numbers are not signed numbers (Sec. 2.4). But a complex number does have a negative for which we have the

Definition. The *negative* of any complex number $a + bi$ is $-a - bi$. Thus, $-5i$ is the negative of $5i$ and $4 - 3i$ is the negative of $-4 + 3i$.

We conclude this article with the

Definition. Two complex numbers that differ only in the sign of their imaginary parts are called *conjugate complex numbers.*

Thus, $a + bi$ and $a - bi$ are conjugate complex numbers.

8.3. FUNDAMENTAL OPERATIONS

The four operations of addition, subtraction, multiplication, and division are called the *fundamental operations.* When applied to complex numbers, these four operations are defined in such a way that they obey all the laws of algebra, as discussed in Chapter 2 for real numbers, with two exceptions. One exception has already been noted, namely, that $i^2 = -1$, a property not possessed by real numbers. The other exception is the following law for real numbers, namely,

$$\text{For } a > 0 \text{ and } b > 0,\ \sqrt{a} \cdot \sqrt{b} = \sqrt{ab}.$$

This law does not apply to imaginary numbers. Thus,

$$\text{for } a > 0 \text{ and } b > 0,\ \sqrt{-a} \cdot \sqrt{-b} \neq \sqrt{(-a)(-b)} = \sqrt{ab}.$$

The correct result is obtained as follows:

$$\sqrt{-a} \cdot \sqrt{-b} = (\sqrt{a}i)(\sqrt{b}i) = i^2\sqrt{ab} = -\sqrt{ab}.$$

To avoid this error we will always write complex numbers in the form $a + bi$, sometimes called the *standard form*, and then operate with i as with any other letter, finally replacing any powers of i as follows: $i^2 = -1$, $i^3 = i^2 \cdot i = -i$, $i^4 = (i^2)^2 = (-1)^2 = 1$, $i^5 = i^4 \cdot i = i$, and so on.

In accordance with the previous discussion, we will now frame definitions for the four fundamental operations, using the two complex numbers $a + bi$ and $c + di$ with the understanding that the final result will also be expressed in the standard form of a complex number.

(1) *Addition.* To add two (or more) complex numbers, we simply add the real and imaginary parts separately, just the same as we combine similar terms in the addition of ordinary algebraic expressions (Sec. 2.4). Thus,

$$(a + bi) + (c + di) = a + c + bi + di,$$

or

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$

and this last statement is the *definition of the sum of two complex numbers.*

(2) *Subtraction.* To subtract one complex number from another, we simply subtract the real and imaginary parts separately. Thus,

$$(a + bi) - (c + di) = a - c + bi - di,$$

or

$$(a + bi) - (c + di) = (a - c) + (b - d)i,$$

and this last statement is the *definition of the difference of two complex numbers.*

(3) *Multiplication.* The product of two complex numbers is obtained by multiplying them together as though they were ordinary binomials. and then replacing i^2 by -1. Thus,

$$(a + bi)(c + di) = ac + adi + bci + bdi^2,$$

or

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i,$$

and this last statement is the *definition of the product of two complex numbers.*

(4) *Division.* To express the quotient of two complex numbers as a single complex number, we use a device analogous to the rationalization of the radical denominator of a fraction (Sec. 2.14). In this case we use the conjugate of the denominator instead of a rationalizing factor. Thus,

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di}$$

$$= \frac{ac - adi + bci - bdi^2}{c^2 - d^2i^2} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2},$$

or

$$\frac{a + bi}{c + di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}\, i, \qquad c + di \neq 0,$$

and this last statement is the *definition of the quotient of two complex numbers.*

In performing the fundamental operations with complex numbers, the student is advised not to use the above definitions as formulas. Instead, he should use the methods employed in obtaining these definitions, as shown in the following examples.

Example 1. For each of the following expressions, perform the indicated operation and express the result in standard form:

(a) $3 + 2\sqrt{-2} - 2(\sqrt{-3} - 1) + 2i - 4$.

(b) $(2 + 3i)(2 - 3i)(1 + 2i)$.

SOLUTION. (a) Whenever necessary we first express any imaginary terms in the form bi. Thus,

$$3 + 2\sqrt{-2} - 2(\sqrt{-3} - 1) + 2i - 4 = 3 + 2\sqrt{2}i - 2(\sqrt{3}i - 1) + 2i - 4 = 3 + 2\sqrt{2}i - 2\sqrt{3}i + 2 + 2i - 4 = (3 + 2 - 4) + (2\sqrt{2} - 2\sqrt{3} + 2)i = 1 + (2\sqrt{2} - 2\sqrt{3} + 2)i.$$

(b) Here the first two factors form a special product (Sec. 2.6) and we write

$$(2 + 3i)(2 - 3i)(1 + 2i) = (4 - 9i^2)(1 + 2i) = (4 - 9[-1])(1 + 2i)$$
$$= 13(1 + 2i) = 13 + 26i.$$

Example 2. Using the binomial theorem, evaluate $(\sqrt{3} - i)^6$ and express the result in standard form.

SOLUTION. In expanding by the binomial theorem, we treat i like any ordinary letter and then finally replace the various powers of i by their simplified values. Thus,

$$(3^{1/2} - i)^6 = (3^{1/2})^6 + 6(3^{1/2})^5(-i) + 15(3^{1/2})^4(-i)^2 + 20(3^{1/2})^3(-i)^3 + 15(3^{1/2})^2(-i)^4 + 6(3^{1/2})(-i)^5 + (-i)^6 = 3^3 - 6 \cdot 3^2\sqrt{3}i + 15 \cdot 3^2 i^2 - 20 \cdot 3\sqrt{3}i^3 + 15 \cdot 3i^4 - 6\sqrt{3}i^5 + i^6 = 27 - 54\sqrt{3}i - 135 + 60\sqrt{3}i + 45 - 6\sqrt{3}i - 1 = (27 - 135 + 45 - 1) + (-54\sqrt{3} + 60\sqrt{3} - 6\sqrt{3})i = -64.$$

Note that in simplifying we take $i^5 = i^4 \cdot i = i$ and $i^6 = i^4 \cdot i^2 = i^2 = -1$. We note also from the result that $\sqrt{3} - i$ is a sixth root of -64.

Example 3. Express $\dfrac{1 + i}{1 - i} - \dfrac{2 - i}{2 + 2i}$ in the standard form of a complex number.

SOLUTION. Here we treat each fraction separately. As in the derivation of the definition of the quotient of two complex numbers, we multiply numerator and denominator by the conjugate of the denominator. Thus,

$$\frac{1+i}{1-i} = \frac{(1+i)(1+i)}{(1-i)(1+i)} = \frac{1+2i+i^2}{1-i^2} = \frac{2i}{2} = i.$$

$$\frac{2-i}{2+2i} = \frac{(2-i)(2-2i)}{(2+2i)(2-2i)} = \frac{4-4i-2i+2i^2}{4-4i^2} = \frac{2-6i}{8} = \frac{1-3i}{4}$$

$$= \frac{1}{4} - \frac{3}{4}i.$$

Hence, $$\frac{1+i}{1-i} - \frac{2-i}{2+2i} = i - \left(\frac{1}{4} - \frac{3}{4}i\right) = -\frac{1}{4} + \frac{7}{4}i.$$

EXERCISES. GROUP 26

In each of Exs. 1–8, find real values of x and y such that the given relation holds.

1. $x + yi = 2 - 3i$.
2. $3x - 2yi = 6 + 4i$.
3. $x + 3y + (2x - 3y - 9)i = 0$.
4. $2x - y + (3y - 2x)i = 2 - 2i$.
5. $(x + yi)^2 = 3 - 4i$.
6. $(x - yi)^2 = -8 - 6i$.
7. $x^2 - 4y + (2y - x)i = 2 - i$.
8. $x^2 + y^2 - 2 + (x + 3y - 2)i = 0$.

In each of Exs. 9–34, perform the indicated operation(s) and express the result in standard form.

9. $(1 + i) + (3 - 2i)$.
10. $(4 - 5i) + (2 + 7i)$.
11. $(2 + \sqrt{-4}) - (3 - \sqrt{-9})$.
12. $(3 + 2i) - (6 - 4i)$.
13. $\sqrt{-4} - \sqrt{-9} + \sqrt{-16}$.
14. $2\sqrt{-36} - \sqrt{-49} + 7$.
15. $\sqrt{-a^2} + \frac{1}{2}\sqrt{-4a^2} - \frac{1}{3}\sqrt{-9a^2}$.
16. $\frac{1}{2}\sqrt{-16a^2} + \frac{1}{a}\sqrt{-4a^4} - \sqrt[3]{-27}$.
17. $(3 + 2i)(3 - 2i)$.
18. $(4 - 3i)(3 + 4i)$.
19. $(1 + i)(1 - 2i)(1 + 3i)$.
20. $(3 - i)(2 + i)(7 - i)$.
21. $(\sqrt{-3} + \sqrt{-2} - \sqrt{-1})(\sqrt{-3} + \sqrt{-2} + \sqrt{-1})$.
22. $(\sqrt{-1} + \sqrt{-2} - \sqrt{-3})(\sqrt{-1} - \sqrt{-2} + \sqrt{-3})$.
23. $(1 - i)^4$.
24. $\left(-\frac{3}{2} + \frac{3}{2}\sqrt{3}i\right)^3$.
25. $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^6$.
26. $\dfrac{5}{\sqrt{-3}}$.
27. $\dfrac{1}{1 - 2i}$.
28. $\dfrac{3}{2 - i}$.
29. $\dfrac{3 - i}{1 + i}$.
30. $\dfrac{2 - i}{1 + 2i}$.
31. $\dfrac{i^5 + 3}{i^3 - 1}$.
32. $(1 - 2i)^{-2}$.
33. $(1 + i)^{-1} - i^{-1}$.
34. $(1 + i)^{-2} - i^{-2}$.

35. Show that the complex number $1 + \sqrt{3}i$ is a root of the equation $2x^4 - 7x^3 + 12x^2 - 8x - 8 = 0$.

36. Show that the complex number $1 - \sqrt{3}i$ is also a root of the equation of Ex. 35.

37. Show that each of the complex numbers $-\frac{1}{2} + \sqrt{\frac{3}{2}}\, i$ and $-\frac{1}{2} - \sqrt{\frac{3}{2}}\, i$ is a cube root of unity.

38. Show that either of the complex cube roots of unity, as given in Ex. 37, is equal to the square of the other.

39. By factoring, find the four roots of the equation $x^4 - 16 = 0$ and show that their sum is equal to zero.

40. Prove that the complex number $a + bi$ is equal to zero if and only if $a = 0$ and $b = 0$.

41. Prove that the sum of any complex number and its negative is equal to zero.

42. Prove that the operation of subtracting one complex number z_1 from another complex number z_2 is equivalent to the operation of adding z_2 and the negative of z_1.

43. If n and k are positive integers such that $n = 4k + m$, where $m = 1, 2$, or 3, show that $i^n = i^m$.

44. If a and b are both positive numbers, show that $\pm\sqrt{a} \cdot \sqrt{-b} = \pm\sqrt{ab}i$, $(-\sqrt{a})(-\sqrt{-b}) = \sqrt{ab}i$, $\sqrt{-a}\,(-\sqrt{-b}) = \sqrt{ab}$.

45. For the two pure imaginary numbers bi and di, obtain definitions for their sum, difference, product, and quotient which are analogous to those given for the complex numbers $a + bi$ and $c + di$ (Sec. 8.3).

46. If the complex number $c + di \neq 0$, show that $c^2 + d^2 \neq 0$ and hence that the result is valid in the definition of the quotient of two complex numbers (Sec. 8.3).

47. Prove that the conjugate of the sum of two complex numbers is equal to the sum of their conjugates.

48. Prove that the conjugate of the product of two complex numbers is equal to the product of their conjugates.

49. Prove that the sum and the product of two conjugate complex numbers are real numbers and that their difference is a pure imaginary number.

50. Prove that if the sum and the product of two complex numbers are both real numbers, they are conjugate complex numbers.

8.4. RECTANGULAR REPRESENTATION

We have seen previously that the real numbers may be represented graphically as points on a straight line (Sec. 3.7). To represent the complex number $x + yi$, however, it will be necessary to make provision for *both* the real numbers x and the pure imaginary numbers yi. This may be done by using the rectangular coordinate system (Sec. 3.8) and assigning the X-axis to the real numbers and the Y-axis to the pure imaginary numbers. Thus, as shown in Fig. 25, the complex number $x + yi$ is represented graphically by the point P, which is x units from the Y-axis and y units from the X-axis. Since the convention of signs for the rectangular coordinate system is to prevail here, the point P has as its rectangular coordinates the *real* number pair (x, y). It is on this basis that we obtain

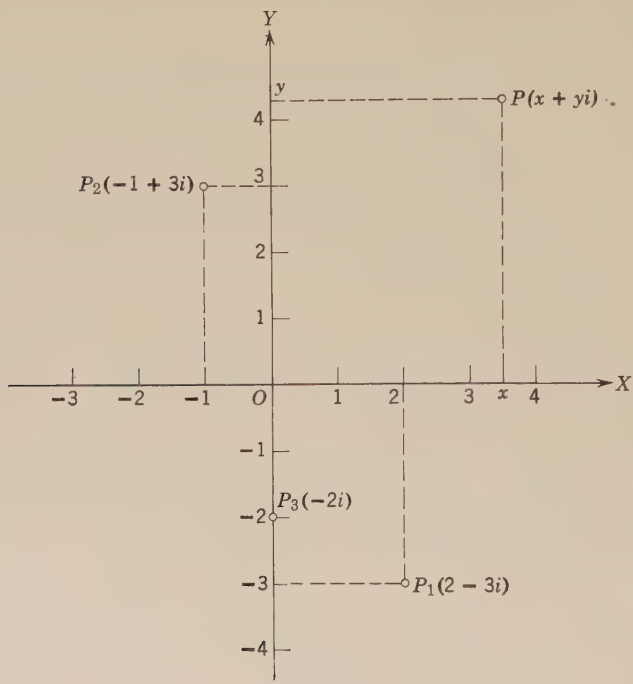


Figure 25

the points P_1, P_2, P_3 representing, respectively, the complex numbers $2 - 3i$, $-1 + 3i$, $-2i$, also shown in Fig. 25. It is customary to refer to the X-axis as the *axis of reals* and to the Y-axis as the *axis of imaginaries*.

In view of this rectangular representation, the complex number $x + yi$, which we have previously called the standard form of the complex number (Sec. 8.3), is more generally referred to as the *rectangular form*. This latter term is particularly convenient when we contrast the rectangular representation of a complex number with its polar representation, which is discussed in the next section.

NOTE 1. Since real numbers are distinct from pure imaginary numbers, it appears logical to represent them graphically on distinct lines. But it is not apparent why these lines should be at right angles to each other as the X- and Y-axes are in the rectangular coordinate system. We will now justify this procedure.

Consider, as shown in Fig. 26, a directed line segment OA along the positive X-axis and of arbitrary unit length, so that the point A represents the positive integer 1. We next introduce an operator, designated by the letter j, having the property that, when multiplied into any directed line segment, it rotates that segment about O in the counterclockwise direction through an angle of 90° but does not change the length of the segment. Hence, if we multiply the directed line segment OA by j, we rotate OA about O through 90° in the counterclockwise direction so that it now occupies the position OB on the positive Y-axis, the point B representing the quantity $j \times 1 = j$. Similarly, applying j to OB, we obtain the directed line segment OC on the negative X-axis, the point C representing the quantity $j \times j = j^2$. Similarly, applying j to OC, we obtain OD on the negative Y-axis, the point D representing the quantity $j \times j^2 = j^3$. Finally, applying j to OD, we arrive at the initial position OA, so that the point A now also represents the quantity $j \times j^3 = j^4$.

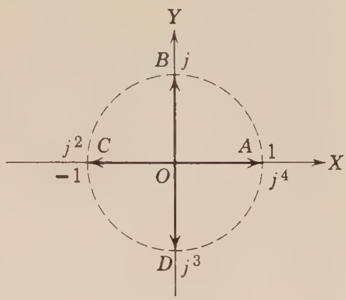


Figure 26

We are now in a position to determine the nature of the operator j by considering the directed unit line segment OA in its various positions on the coordinate axes. Since A represents 1 on the X-axis, C represents -1, that is, $j^2 = -1$. Similarly, since B represents j on the Y-axis, D represents $-j$, that is, $j^3 = -j$. Also, for the point A, $j^4 = 1$. But all of these relations are precisely the properties of the imaginary unit i. Hence the operator j and the imaginary unit i are identical, and this explains why the pure imaginary numbers are represented by points on the Y-axis.

We next consider the *graphical representation of the sum of two given complex numbers*. Let the points $P_1(a, b)$ and $P_2(c, d)$ represent the complex numbers $a + bi$ and $c + di$, respectively, as shown in Fig. 27. Connect each of these points with the origin O and complete the parallelogram OP_1PP_2 having OP_1 and OP_2 as adjacent sides. Let A, B, and C, respectively, be the feet of the perpendiculars drawn from P_2, P_1, and P to the X-axis, and draw P_1D perpendicular to PC. By geometry, the right triangles OP_2A and P_1PD are congruent so that $OA = P_1D = BC$ and $AP_2 = DP$. Then

$$OC = OB + BC = OB + P_1D = OB + OA = a + c,$$
$$CP = CD + DP = BP_1 + AP_2 = b + d.$$

Hence the point P represents the complex number $(a + c) + (b + d)i$, the *sum* of the two complex numbers $a + bi$ and $c + di$.

To *subtract* one complex number from another graphically, say $c + di$ from $a + bi$, we add the complex numbers $a + bi$ and $-c - di$, using the graphical method for addition described above. It is left to the student as an exercise to draw a figure, similar to Fig. 27, representing the *difference* of two given complex numbers.

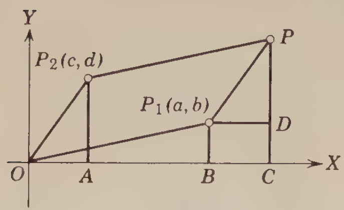


Figure 27

NOTE 2. The student who has studied the addition of two vectors in physics by means of a parallelogram will recognize that the operation is identical with that of the graphical representation of the addition of two complex numbers. It appears then that complex numbers and vectors are closely related. We shall comment further on this later (Sec. 8.8).

NOTE 3. The graphical representation of the product and also the quotient of two given complex numbers in rectangular coordinates may be effected by means of geometric constructions. However, as shown in the next section, these operations may be very conveniently considered by means of another representation known as the polar form of a complex number.

8.5. POLAR REPRESENTATION

We now introduce a trigonometric form of the complex number, a form which has certain distinct advantages over the rectangular form. In Fig. 28, let the point P represent the complex number $x + yi$. Draw the line segment OP connecting P and the origin, and let its length be denoted by r. From P drop the perpendicular PA to the X-axis, and denote the angle POA by θ. Then by trigonometry (Appendix I), we have from the right triangle OAP,

(1) $\quad x = r\cos\theta, \qquad y = r\sin\theta,$

(2) $\quad r = \sqrt{x^2 + y^2}, \qquad r \geq 0,$

(3) $\quad \tan\theta = \dfrac{y}{x}, \qquad x \neq 0.$

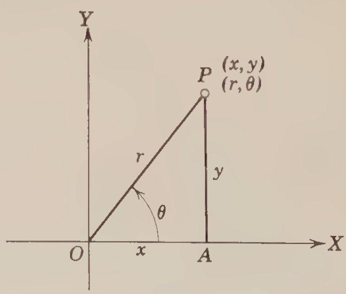


Figure 28

Hence, from relation (1) we may write

(4) $\quad x + yi = r(\cos\theta + i\sin\theta).$

The right member of (4) is called the *polar form* of the complex number. The length r is called the *modulus* or *absolute value* of the complex number and is always a non-negative quantity whose value is given by (2). The angle θ is called the *amplitude* or *argument* of the complex number and, unless otherwise specified, will be restricted to the range $0 \leq \theta < 360°$.

NOTE 1. Since the modulus r is also called the *absolute value* (Sec. 2.4) of the complex number, we may write $r = |x + iy|$.

For a particular complex number, the amplitude θ has a unique value which is non-negative and less than 360°, and may be determined from the relations (1). It may also be determined from relation (3) and the quadrant in which θ lies. We note that relation (3) has the restriction that $x \neq 0$. If $x = 0$, the complex number $x + yi$ takes the form of the pure imaginary number so that $\theta = 90°$ if $y > 0$ and $\theta = 270°$ if $y < 0$. It is clear that a complex number and its graphical representation are uniquely determined for given values of r and θ.

In this and the following section we shall consider the operations of multiplication, division, involution, and evolution on complex numbers when in the polar form. Hence, if the complex numbers are given in rectangular form, it is extremely important to obtain their correct polar forms. The process is illustrated in

Example 1. Find the modulus, amplitude, and polar form of the complex number $-2 + 2i$.

SOLUTION. In order to reduce the possibility of error, it is always best first to plot the given complex number, as shown in Fig. 29. Then the modulus is given by

$$r = \sqrt{x^2 + y^2} = \sqrt{4 + 4} = 2\sqrt{2}.$$

For the amplitude θ we have

$$\tan \theta = y/x = 2/-2 = -1,$$

from which, since θ is in the second quadrant, the amplitude $\theta = 135°$. Hence the polar form of $-2 + 2i$ is

$$2\sqrt{2}(\cos 135° + i \sin 135°).$$

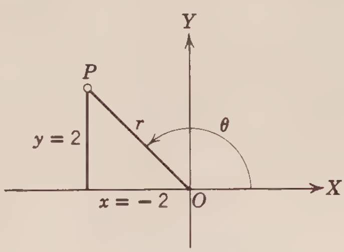


Figure 29

As a check, we may evaluate the polar form and see if we obtain the given rectangular form. This is left as an exercise to the student.

We now consider the product of two given complex numbers in polar form. Thus,

$$\begin{aligned}
&[r_1(\cos \theta_1 + i \sin \theta_1)][r_2(\cos \theta_2 + i \sin \theta_2)] \\
&\quad = r_1r_2(\cos \theta_1 \cos \theta_2 + i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \cos \theta_2 + i^2 \sin \theta_1 \sin \theta_2) \\
&\quad = r_1r_2[(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)] \\
&\quad = r_1r_2[\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)],
\end{aligned}$$

by the addition formulas of trigonometry (Appendix I).

We state this result as

Theorem 1. *The modulus of the product of two complex numbers is equal to the product of their moduli and the amplitude of the product is equal to the sum of their amplitudes.*

Corollary. *The modulus of the product of three or more complex numbers is equal to the product of the moduli of the factors and the amplitude of the product is equal to the sum of the amplitudes of the factors.*

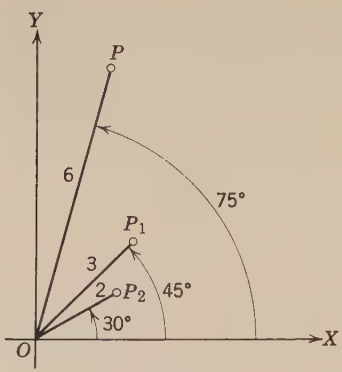


Figure 30

Example 2. Find the product of the complex numbers $3(\cos 45° + i \sin 45°)$ and $2(\cos 30° + i \sin 30°)$. Illustrate the process graphically.

SOLUTION. By Theorem 1, the modulus of the product $= 2 \cdot 3 = 6$, and the amplitude $= 45° + 30° = 75°$. Hence the product in polar form is the complex number $6(\cos 75° + i \sin 75°)$.

The results are shown in Fig. 30 where the points P_1, P_2, and P represent the first factor, second factor, and product, respectively.

We next consider the quotient of two given complex numbers in polar form. Thus,

$$\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2} \cdot \frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2} \cdot \frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}$$

$$= \frac{r_1}{r_2} \cdot \frac{\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - i\sin\theta_1\sin\theta_2}{\cos^2\theta_2 - i^2\sin^2\theta_2}$$

$$= \frac{r_1}{r_2} \cdot \frac{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2 = 1}$$

$$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)],$$

by the subtraction formulas of trigonometry (Appendix I).

We state this result as

Theorem 2. *The modulus of the quotient of two complex numbers is equal to the modulus of the dividend divided by the modulus of the divisor, and the amplitude of the quotient is equal to the amplitude of the dividend minus the amplitude of the divisor.*

Example 3. Find the indicated quotient and express the result in rectangular form:

$$4(\cos 75° + i \sin 75°) \div 2(\cos 45° + i \sin 45°).$$

SOLUTION. By Theorem 2,

$$\frac{4(\cos 75° + i \sin 75°)}{2(\cos 45° + i \sin 45°)} = \frac{4}{2}[\cos (75° - 45°) + i \sin (75° - 45°)]$$
$$= 2(\cos 30° + i \sin 30°) = \sqrt{3} + i.$$

The student should illustrate this problem graphically.

NOTE 2. If the amplitude of a complex number is a special angle such as 30° or 45°, or a multiple of either of these angles, the polar form is readily transformed into the rectangular form, and vice versa. But for other angles, a table of natural trigonometric functions is required (Appendix II).

EXERCISES. GROUP 27

In each of Exs. 1–9 plot the given complex number, its conjugate, and its negative.

1. $1 + 3i$.
2. $-2 + 2i$.
3. $-1 - 2i$.
4. $4 - 2i$.
5. $3i$.
6. $5 + \sqrt{-4}$.
7. $\sqrt{-9} + 1$.
8. -3.
9. $2i - 7$.

In each of Exs. 10–23, perform the indicated operation(s) both algebraically and graphically.

10. $(1 - i) + (2 + 3i)$.
11. $(3 + 2i) + (-2 - i)$.
12. $(-2 - \sqrt{-4}) + (5 - 2i)$.
13. $(4 + \sqrt{-9}) + (1 - \sqrt{-16})$.
14. $(-1 + 2i) - (2 - 3i)$.
15. $(3i + 2) - (3 + 2i)$.
16. $(6 + \sqrt{-9}) - (3 - \sqrt{-4})$.
17. $(3 + 2i) + 5$.
18. $(3 + 2i) - 5$.
19. $(2 - 7i) + 4i$.
20. $(5 + i) + (-3 - 2i) + (1 + 3i)$.
21. $(2 - 4i) + (6 + i) + (-7 - i)$.
22. $(8 + i) + (1 - 3i) - (6 - 2i)$.
23. $(4 - 2i) - (2 + i) + (-2 - i)$.

In each of Exs. 24–32, find the modulus, amplitude, and polar form of the given complex number.

24. $1 + i$.
25. $-2 + 2\sqrt{3}i$.
26. $3 - 3\sqrt{3}i$.
27. $-\sqrt{3} - i$.
28. $-\sqrt{2} + \sqrt{2}i$.
29. -7.
30. $2\sqrt{2} - 2\sqrt{2}i$.
31. $-4 - 4\sqrt{3}i$.
32. $3i$.

In each of Exs. 33–36, find the indicated product, using Theorem 1 of Sec. 8.5. Express the result in rectangular form.

33. $2(\cos 30° + i \sin 30°) \cdot 3(\cos 60° + i \sin 60°)$.
34. $3(\cos 45° + i \sin 45°) \cdot \sqrt{2}(\cos 90° + i \sin 90°)$.
35. $4(\cos 180° + i \sin 180°) \cdot \frac{1}{2}(\cos 30° + i \sin 30°)$.
36. $(\cos 20° + i \sin 20°) \cdot 4(\cos 100° + i \sin 100°)$.

In each of Exs. 37–40, find the indicated quotient, using Theorem 2 of Sec. 8.5. Express the result in rectangular form.

37. $\dfrac{3(\cos 130° + i \sin 130°)}{2(\cos 70° + i \sin 70°)}$.
38. $\dfrac{5(\cos 135° + i \sin 135°)}{\cos 45° + i \sin 45°}$.
39. $\dfrac{6(\cos 220° + i \sin 220°)}{3(\cos 40° + i \sin 40°)}$.
40. $\dfrac{4(\cos 70° + i \sin 70°)}{2(\cos 50° + i \sin 50°)}$.

41. Show how the method of finding the sum of two given complex numbers graphically may be extended to the sum of three or more complex numbers.

42. Draw a figure illustrating the graphical method of finding the difference of two given complex numbers. Explain each step fully, as in the analogous problem of graphical addition (Sec. 8.4).

43. If the point P_1 represents a complex number and the point P_2 represents the negative of that number, show that the line segment P_1P_2 passes through the origin O and is bisected at O.

44. Show that if a complex number is equal to zero, its modulus is equal to zero, and conversely.

45. Show that a complex number and its negative have the same modulus.

46. Show that a complex number and its conjugate have the same modulus.

47. Show algebraically that the modulus of the product of two complex numbers is equal to the product of their moduli.

48. Show algebraically that the modulus of the quotient of two complex numbers is equal to the quotient of their moduli.

49. Show that if two complex numbers are equal, their moduli are equal but that the converse is not necessarily true.

50. Establish the corollary to Theorem 1 (Sec. 8.5).

51. By multiplying any complex number in polar form by the imaginary unit i in polar form, show that the amplitude of the product exceeds that of the given complex number by 90°. Compare this result with the definition of the operator j in Sec. 8.4.

In each of Exs. 52–55, z_1 and z_2 represent, respectively, the complex numbers $x_1 + y_1 i$ and $x_2 + y_2 i$.

52. Show graphically that the modulus or absolute value of the sum of two complex numbers is less than or equal to the sum of their moduli or absolute values, that is, show that

$$|z_1 + z_2| \le |z_1| + |z_2|.$$

Hint: The sum of two sides of a triangle is greater than the third side.

53. Show graphically that the modulus or absolute value of the difference of two complex numbers is greater than or equal to the difference of their moduli or absolute values, that is, show that

$$|z_1 - z_2| \ge |z_1| - |z_2|.$$

54. Establish the result of Ex. 52 algebraically. (See Example 2 of Sec. 6.6.)

55. Establish the result of Ex. 53 algebraically. (See Ex. 11 of Group 23, Sec. 6.6.)

8.6. INVOLUTION AND EVOLUTION

We now consider the remaining two algebraic operations, involution and evolution, applied to complex numbers. Since involution is a special case of multiplication (Sec. 1.3), we first refer to Theorem 1 of Sec. 8.5 on the multiplication of two complex numbers. It follows from this theorem that if the two complex numbers are each equal to $r(\cos \theta + i \sin \theta)$, then their product is given by the relation

$$[r(\cos \theta + i \sin \theta)]^2 = r^2(\cos 2\theta + i \sin 2\theta).$$

As an immediate extension of this result, we have

$$[r(\cos \theta + i \sin \theta)]^3 = r^3(\cos 3\theta + i \sin 3\theta).$$

We are thus led to infer that if n is any positive integer, then

$$(1) \qquad [r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta).$$

Relation (1), known as *De Moivre's theorem*, may be established by mathematical induction (Sec. 7.2) as we shall now show.

The relation is obviously true for $n = 1$. Assuming it is true for $n = k$, we have the assumed equality

$$(2) \qquad [r(\cos \theta + i \sin \theta)]^k = r^k(\cos k\theta + i \sin k\theta).$$

Multiplying both sides of (2) by $r(\cos \theta + i \sin \theta)$, we have

$$(3) \qquad [r(\cos \theta + i \sin \theta)]^{k+1} = r^{k+1}[\cos (k + 1)\theta + i \sin (k + 1)\theta],$$

where the right side of (3) follows from Theorem 1 of Sec. 8.5.

But relation (3), which follows directly from relation (2), is precisely the same as relation (1) with n replaced by $k + 1$. We have therefore shown that if (1) is true for $n = k$, it is also true for $n = k + 1$. Hence, since (1) is true for $n = 1$, it is also true for $n = 2$. Similarly, if it is true for $n = 2$, it is true for $n = 3$, and so on for all positive integral values of n. This completes the proof of

Theorem 3. (*De Moivre's Theorem*). *If n is any positive integer, and if r and θ are, respectively, the modulus and amplitude of any complex number, then*

$$[r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta),$$

that is, if n is a positive integer, the modulus of the nth power of a complex number is equal to the nth power of the modulus of the number, and the amplitude of the nth power is equal to n times the amplitude of the number.

Example 1. Using De Moivre's theorem, evaluate $(\sqrt{3} - i)^6$ and express the result in rectangular form.

SOLUTION. This problem involves exactly the same expansion evaluated by means of the binomial theorem in Example 2 of Sec. 8.3. By comparison, we will now see one of the advantages of the polar form of the complex number. We first express $\sqrt{3} - i$ in the polar form and then apply De Moivre's theorem. Thus,

$$
\begin{aligned}
(\sqrt{3} - i)^6 &= [2(\cos 330° + i \sin 330°)]^6, \\
\text{by De Moivre's theorem,} \quad &= 2^6(\cos 1980° + i \sin 1980°), \\
\text{by trigonometry,} \quad &= 64(\cos 180° + i \sin 180°), \\
&= 64(-1 + 0) = -64.
\end{aligned}
$$

We next consider the operation of evolution, that is, the determination of the roots of a complex number. For n a positive integer, let r be a positive number and $r^{1/n}$ its principal nth root and therefore also a unique positive number (Sec. 2.13). Consider a complex number with modulus $r^{1/n}$ and amplitude θ/n so that its polar form is $r^{1/n}\left(\cos \frac{\theta}{n} + i \sin \frac{\theta}{n}\right)$. The nth power of this number is $r(\cos \theta + i \sin \theta)$ by Theorem 3 (De Moivre's theorem), that is,

$$r(\cos \theta + i \sin \theta) = \left[r^{1/n}\left(\cos \frac{\theta}{n} + i \sin \frac{\theta}{n}\right)\right]^n.$$

Taking the nth root of both sides of this last relation, we have

$$(4) \qquad [r(\cos \theta + i \sin \theta)]^{1/n} = r^{1/n}\left(\cos \frac{\theta}{n} + i \sin \frac{\theta}{n}\right),$$

that is, De Moivre's theorem also holds for $1/n$, the reciprocal of any positive integer.

Relation (4), as it stands, gives only one nth root of a complex number. We will now see how to obtain all of the nth roots. The values of the trigonometric functions of any angle remain unaltered if that angle is increased or decreased by any positive integral multiple of 360°. Hence, for any complex number, if k is a non-negative integer, we may write

$$r(\cos \theta + i \sin \theta) = r[\cos (\theta + k \cdot 360°) + i \sin (\theta + k \cdot 360°)],$$

where the right member is often called the *general* or *complete polar form* of a complex number. Taking the nth root of both sides in accordance with relation (4), we have

$$(5) \quad [r(\cos \theta + i \sin \theta)]^{1/n} = r^{1/n}\left[\cos \frac{\theta + k \cdot 360°}{n} + i \sin \frac{\theta + k \cdot 360°}{n}\right].$$

If in (5) we set $k = 0, 1, 2, 3, \cdots, n - 1$ in turn we obtain the following n distinct nth roots of $r(\cos\theta + i\sin\theta)$:

For $k = 0$, $$r^{1/n}\left[\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right],$$

$k = 1$, $$r^{1/n}\left[\cos\frac{\theta + 360°}{n} + i\sin\frac{\theta + 360°}{n}\right],$$

$k = 2$, $$r^{1/n}\left[\cos\frac{\theta + 2\cdot 360°}{n} + i\sin\frac{\theta + 2\cdot 360°}{n}\right],$$

. . . .

$k = n - 1$, $$r^{1/n}\left[\cos\frac{\theta + (n-1)360°}{n} + i\sin\frac{\theta + (n-1)360°}{n}\right].$$

These n roots are all different because the amplitudes of any two of them differ by less than 360°. Furthermore, there are no more than n distinct roots, for, if we assign values to k greater than $n - 1$, we obtain the same roots over again. Thus, for $k = n$, the root takes the form $r^{1/n}\left[\cos\left(\frac{\theta}{n} + 360°\right) + i\sin\left(\frac{\theta}{n} + 360°\right)\right]$ which is identical with the root obtained for $k = 0$.

We note, furthermore, that since all n roots have the same modulus $r^{1/n}$ and since, for successive values of k, the amplitudes differ by $360°/n$, the graphical representation of these roots consists of points equally spaced on the circumference of a circle whose center is at the origin and whose radius is equal to the common modulus $r^{1/n}$.

The preceding results are summarized in

Theorem 4. *Every number (except zero), real or complex, has exactly n distinct nth roots.*

If the modulus and amplitude of any number are represented by r and θ, respectively, the n roots are given by the expression

$$r^{1/n}\left[\cos\frac{\theta + k\cdot 360°}{n} + i\sin\frac{\theta + k\cdot 360°}{n}\right]$$

where $r^{1/n}$ represents the principal nth root of the positive number r and k takes on the values $0, 1, 2, \cdots, (n - 1)$ *in turn.*

These roots are represented graphically by the vertices of a regular polygon of n sides inscribed in a circle whose center is at the origin and whose radius is equal to $r^{1/n}$.

It has now been shown that De Moivre's theorem is true when the exponent n is any positive integer or the reciprocal of any positive integer.

We may also show that it holds when n is a negative integer and finally when n is any rational number. The proofs of these last two cases are left as exercises to the student.

NOTE 1. De Moivre's theorem holds for *all* values of the exponent n, real or complex. The proof for values of n other than rational values is beyond the scope of this book.

Example 2. Find the four fourth roots of $-8 + 8\sqrt{3}i$ and represent them graphically.

SOLUTION. We first obtain the polar form of the given complex number. Thus,

$$-8 + 8\sqrt{3}i = 16(\cos 120° + i \sin 120°),$$

in general polar form,

$$= 16[\cos (120° + k \cdot 360°) + i \sin (120° + k \cdot 360°)].$$

Then by Theorem 4, the expression for the fourth roots is given by

$$16^{1/4}\left[\cos \frac{120° + k \cdot 360°}{4} + i \sin \frac{120° + k \cdot 360°}{4}\right]$$
$$= 2[\cos (30° + k \cdot 90°) + i \sin (30° + k \cdot 90°)].$$

Assigning to k the successive values 0, 1, 2, 3, we have the four required roots:

$k = 0,$

$$2(\cos 30° + i \sin 30°) = \sqrt{3} + i,$$

$k = 1,$

$$2(\cos 120° + i \sin 120°) = -1 + \sqrt{3}i,$$

$k = 2,$

$$2(\cos 210° + i \sin 210°) = -\sqrt{3} - i,$$

$k = 3,$

$$2(\cos 300° + i \sin 300°) = 1 - \sqrt{3}i.$$

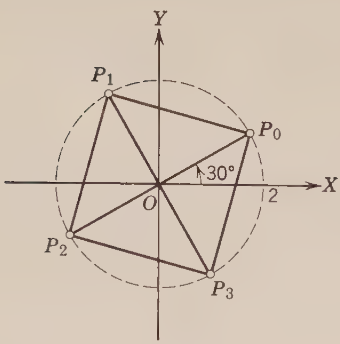


Figure 31

The roots are represented graphically in Fig. 31 by the points P_0, P_1, P_2, P_3, where the subscripts correspond to the values assigned to k. These points lie on the circumference of a circle whose center is the origin O and whose radius is equal to 2, the common modulus of the roots. Furthermore, the points are the vertices of a square inscribed in the circle.

Example 3. Find all the roots of the equation $x^3 - 1 = 0$ by two methods: (a) by means of De Moivre's theorem and (b) algebraically.

SOLUTION. (a) The solution of this equation requires the determination of the three cube roots of unity. Hence we proceed as in the previous example. Thus,

$$1 = 1(\cos 0° + i \sin 0°) = \cos k \cdot 360° + i \sin k \cdot 360°.$$

By Theorem 4, the expression for the cube roots is

$$\cos \frac{k \cdot 360°}{3} + i \sin \frac{k \cdot 360°}{3} = \cos k \cdot 120° + i \sin k \cdot 120°.$$

Then for the three cube roots we have

for $k = 0$, $\qquad \cos 0° + i \sin 0° = 1,$

$k = 1$, $\qquad \cos 120° + i \sin 120° = -\frac{1}{2} + \frac{\sqrt{3}}{2} i,$

$k = 2$, $\qquad \cos 240° + i \sin 240° = -\frac{1}{2} - \frac{\sqrt{3}}{2} i.$

(b) The equation $x^3 - 1 = 0$ may be readily solved by factoring.

Thus, $\qquad (x - 1)(x^2 + x + 1) = 0.$

From the first factor we have the root $x = 1$.

Setting the second factor equal to zero and using the quadratic formula, we have

$$x = \frac{-1 \pm \sqrt{1 - 4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i.$$

NOTE 2. We have now seen that if we apply any of the six operations of algebra to complex numbers, the result is always a complex number, that is, the complex number system is sufficient for our algebra. In this connection the student should reread the last paragraph of Sec. 1.4.

EXERCISES. GROUP 28

In the exercises of this group, if the amplitudes are special angles whose trigonometric functions may be found without using tables, the final results should be expressed in rectangular form; otherwise they may be left in polar form.

In each of Exs. 1–12, find the indicated power by De Moivre's theorem.

1. $[2(\cos 15° + i \sin 15°)]^3$.
2. $[\sqrt{2}(\cos 30° + i \sin 30°)]^4$.
3. $[\sqrt{3}(\cos 15° + i \sin 15°)]^6$.
4. $[2(\cos 45° + i \sin 45°)]^4$.
5. $[\sqrt{5}(\cos 20° + i \sin 20°)]^4$.
6. $[2^{1/4}(\cos 150° + i \sin 150°)]^8$.
7. $(1 + i)^6$.
8. $(-1 + \sqrt{3}i)^4$.
9. $(1 - i)^{-3}$.
10. $(-\sqrt{\tfrac{3}{2}} - \tfrac{1}{2}i)^7$.
11. $(-\sqrt{\tfrac{3}{2}} + \tfrac{1}{2}i)^9$.
12. $(-\sqrt{\tfrac{2}{2}} - \sqrt{\tfrac{2}{2}}i)^{12}$.

In each of Exs. 13–18, find the indicated power by (a) the binomial theorem; (b) De Moivre's theorem.

13. $(1 - \sqrt{3}i)^3$. 14. $(-1 + i)^4$. 15. $(\sqrt{3} - i)^8$.
16. $(\frac{1}{2} + \frac{\sqrt{3}}{2}i)^7$. 17. $(-1 + \sqrt{3}i)^5$. 18. $(-1 - \sqrt{3}i)^6$.

In each of Exs. 19–31, find the specified roots and represent them graphically.

19. The three cube roots of -27.
20. The three cube roots of $8(\cos 60° + i \sin 60°)$.
21. The three cube roots of $-2 + 2i$.
22. The four fourth roots of $-8 - 8\sqrt{3}i$.
23. The four fourth roots of -4.
24. The four fourth roots of $4 - 4\sqrt{3}i$.
25. The five fifth roots of 32.
26. The five fifth roots of $-16 - 16\sqrt{3}i$.
27. The six sixth roots of $27i$.
28. The six sixth roots of $1 + \sqrt{3}i$.
29. The eight eighth roots of $-128 + 128\sqrt{3}i$.
30. The eight eighth roots of $-\frac{1}{2} - \dfrac{\sqrt{3}}{2} i$.
31. The nine ninth roots of $-i$.

In each of Exs. 32–37, find all the roots of the given equation by means of De Moivre's theorem and also algebraically.

32. $x^3 + 8 = 0$. 33. $x^6 - 1 = 0$. 34. $x^6 - 64 = 0$.
35. $x^4 - 1 = 0$. 36. $x^4 - 16 = 0$. 37. $x^3 - 27 = 0$.

38. In the statement of Theorem 4 (Sec. 8.6), the number zero is excluded. Give the reason for this.

39. Show that De Moivre's theorem holds when $n = -m$, a negative integer.

40. Show that De Moivre's theorem holds when $n = p/q$, any rational number.

8.7. GROUPS

In this section there is given a brief and elementary introduction to a topic of great importance in more advanced mathematics, namely, the concept of a group for which we have the following

Definition. A set of elements is said to form a *group* with respect to a single operation (denoted by the symbol o) provided that these elements obey the following four postulates:

1. *Closure.* If a and b are any two elements (not necessarily distinct) of the set, then $a \circ b$ is a unique element of the set.

2. *Associative.* If a, b, and c are any three elements of the set, then $(a \circ b) \circ c = a \circ (b \circ c)$.

3. *Identity.* There exists an element e in the set, called the *identity element*, having the property that, for every element a in the set,

$$a \circ e = e \circ a = a.$$

4. *Inverse.* For every element a in the set, there exists an element a' also in the set such that

$$a \circ a' = a' \circ a = e.$$

The element a' is called the *inverse* of a.

As an indication of the importance of the group concept in the fields of analysis and geometry it may be noted that the elements of a group may not only be the ordinary numbers of our algebra but also such varied entities as matrices, quaternions, vectors, substitutions, and transformations.

A simple example of a group is the set of all positive and negative integers and zero when the law of combination is the operation of addition. Thus, if a and b are any two integers, then $a + b$ is a unique integer (Sec. 2.3), and Postulate 1 is satisfied. Postulate 2 is satisfied since addition is associative (Sec. 2.3). The number zero is the unique identity element since zero is the only number having the property that for any integer a, $a + 0 = 0 + a = a$ (Sec. 2.4). Hence, Postulate 3 is satisfied. Finally, Postulate 4 is satisfied since each integer has its corresponding negative as its inverse, that is, if a is any integer, then $a + (-a) = (-a) + a = 0$ (Sec. 2.4, Theorem 1). Since the number of elements in this group is unlimited, it is called an *infinite group.* An example of a *finite group* is given below.

The reason for introducing the group concept at this point is illustrated in the following

Example. Show that the three cube roots of unity constitute a group with respect to the operation of multiplication.

SOLUTION. In Example 3 of Sec. 8.6, we found that the three cube roots of unity are 1, $-\frac{1}{2} + \frac{\sqrt{3}}{2} i$, and $-\frac{1}{2} - \frac{\sqrt{3}}{2} i$.

It is easy to show that any one of these complex cube roots is equal to the square of the other (Ex. 38, Group 26, Sec. 8.3). Hence, if one complex cube root is denoted by ω, the other may be represented by ω^2. We are to show, therefore, that the three quantities 1, ω, and ω^2 constitute a group with respect to multiplication by proving that they satisfy the four postulates of a group.

1. The product of any two cube roots is also a cube root of unity. Thus, $1 \times \omega = \omega$, $1 \times \omega^2 = \omega^2$, $\omega \times \omega^2 = \omega^3 = 1$.

2. The associative law holds, that is, $(1 \times \omega) \times \omega^2 = 1 \times (\omega \times \omega^2) = \omega^3$.

3. The identity element is obviously 1.
4. The inverse of each element is its reciprocal, and these reciprocals are also elements of the group. Thus,

$$\frac{1}{1} = 1, \qquad \frac{1}{\omega} = \frac{\omega^2}{\omega^3} = \frac{\omega^2}{1} = \omega^2, \qquad \frac{1}{\omega^2} = \frac{\omega}{\omega^3} = \frac{\omega}{1} = \omega.$$

Obviously, the product of any element and its inverse is the identity element 1.

8.8. VECTORS

In this section we consider briefly the subject of vectors, which, as we have previously noted (Sec. 8.4, Note 2), are closely related to complex numbers.

In physics a *vector* is a quantity which has both magnitude and direction. Examples of vectors are force, velocity, and acceleration. A vector may be represented graphically by a directed line segment whose length, according to a suitable scale, denotes the magnitude of the vector. Since we shall here consider only coplanar vectors, that is, vectors lying in the same plane, we will use the plane of the rectangular coordinate system as their common plane. This also gives us a very convenient means for representing vectors in the plane. Thus, as shown in Fig. 32, the line segment $\overline{OP}$ directed *from* the origin O *to* the point P represents a vector whose length $OP = r$ denotes the *magnitude* of the vector. The *direction* of the vector is given by the angle θ which the directed line segment OP makes with the positive X-axis. The arrow head gives the *sense* of the direction and indicates that the vector is directed from its *initial point* O to its *terminal point* P. The projection of the vector on the X-axis, the directed line segment $OA = x$, is called its *horizontal component*, and the projection on the Y-axis, the directed line segment $OB = y$, is called its *vertical component*.

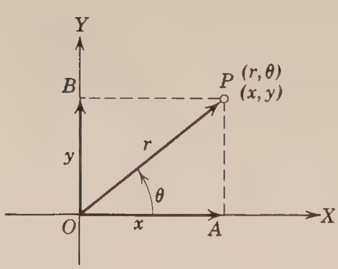


Figure 32

It is now evident from the above orientation of a vector, with its initial point at the origin O, that a vector is completely specified provided that we can definitely locate its terminal point P. But, as we have previously seen, the point P is uniquely determined as the geometric representation of a complex number. In the rectangular form (Sec. 8.4), P represents the complex number $x + yi$ where x is the horizontal component and y the vertical component of a vector whose magnitude is represented by a directed line segment of length $\sqrt{x^2 + y^2}$. In the polar form (Sec. 8.5), P represents the complex number $r(\cos\theta + i\sin\theta)$ where the modulus r represents the magnitude of the vector and the amplitude θ gives the direction of the vector with respect to the positive X-axis. It follows, therefore, that if the initial point of a vector is the origin of the rectangular coordinate system, the vector is completely specified if we know either of the number pairs (x, y) or (r, θ), where the letters have the significance previously stated. Hence it will be convenient to designate a vector by either of these number pairs.

Two vectors are said to be *equal* provided that they are represented by two directed line segments having the same length, the same direction, and the same sense. Thus, any vector a located anywhere in the coordinate plane may be replaced by a vector represented by a line segment parallel and equal in length to the line segment representing a and also having the same sense but with its initial point at the origin. We may then designate the equal vector by either of the number pairs (x, y) or (r, θ).

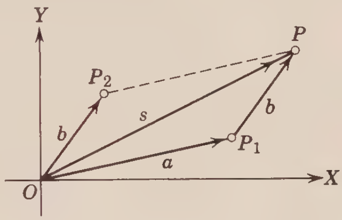


Figure 33

Consider now the vectors a and b having the respective terminal points P_1 and P_2 but the same initial point O, as shown in Fig. 33. Draw the line segment P_1P parallel and equal in length to OP_2 so that the line segment P_1P with its initial point at P_1 also represents the vector b. The point P is then the terminal point of a vector s represented by the directed line segment OP and *defined* as the *sum* of the vectors a and b, that is, $s = a + b$. If the student will now compare this definition with the discussion of Fig. 27 in Sec. 8.4, he will see the similarity between the addition of vectors and the addition of complex numbers. We note also that by drawing the line segment P_2P we complete a parallelogram, the basis of the so-called *parallelogram law* for the addition of two vectors.

Example. Find the sum of the two vectors a (6, 30°) and b (4, 60°) both graphically and analytically.

SOLUTION. For the graphical addition we follow the procedure outlined in the definition of addition above. We first plot the terminal points $P_1(6, 30°)$ and $P_2(4, 60°)$ of the given vectors a and b, respectively, and then complete the parallelogram having OP_1 and OP_2 as adjacent sides, as shown in Fig. 34. This gives us the terminal point P of the required vector sum OP.

By geometry, the horizontal and vertical components of vector a are $3\sqrt{3}$ and 3, respectively, and the horizontal and vertical components of vector b are 2 and $2\sqrt{3}$, respectively. Hence the horizontal and vertical components of the vector sum OP are $3\sqrt{3} + 2$ and $3 + 2\sqrt{3}$, respectively. We then have for the magnitude and direction, respectively, of the vector sum OP:

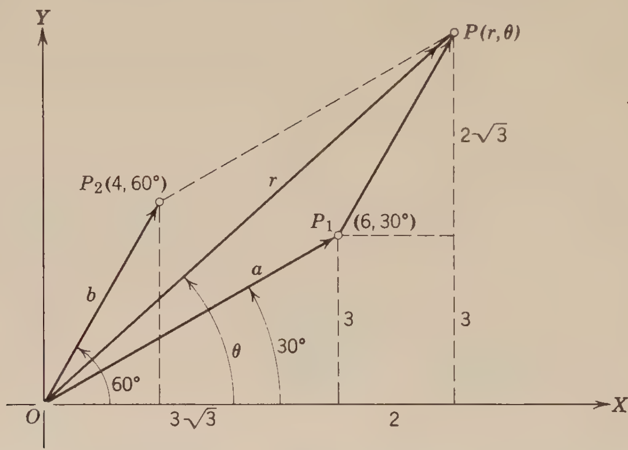


Figure 34

$$r = \sqrt{x^2 + y^2} = \sqrt{(3\sqrt{3} + 2)^2 + (3 + 2\sqrt{3})^2} = 9.673,$$

$$\theta = \text{arc tan}\,\frac{y}{x} = \text{arc tan}\,\frac{3 + 2\sqrt{3}}{3\sqrt{3} + 2} = 41^\circ\ 56'.$$

One vector is said to be the *negative* of another vector if both vectors are parallel, have the same magnitude, but are opposite in sense. To subtract the vector b from the vector a, we add the negative of b to a, that is,

$$a - b = a + (-b),$$

or the difference $a - b$ of two vectors is equal to the sum $a + (-b)$. Hence we may obtain the difference of two vectors by an equivalent operation of addition as described above.

In conclusion, even this brief discussion indicates the close relation between vectors and complex numbers. This relation arises in many applications, for example, in the theory of alternating current circuits.

The properties and applications of vectors comprise a vast field of great importance. They are the subject of advanced study in treatises on vector analysis.

8.9. FUNCTIONS OF A COMPLEX VARIABLE

We close this chapter with a few brief remarks on functions of a complex variable. In Sec. 3.3 we defined y as a function of a single variable x. If x is restricted to real values, we say that y is a *function of a real variable* x. If, however, we have a functional relation in which the independent variable is permitted to assume complex as well as real values, we are said to have a *function of a complex variable*. It is customary to express this relation in the form

(1) $$w = f(z),$$

where $z = x + yi$, x and y are real variables, and i is the imaginary unit. It follows, in general, that w may be written in terms of two expressions containing the variables x and y, one with real coefficients and the other with imaginary coefficients. We write this in the form

(2) $$w = u(x, y) + iv(x, y),$$

where u and v are each functions of the real variables x and y. We illustrate this situation in the following

Example. If $w = z^2$, where $z = x + yi$, find the functions $u(x, y)$ and $v(x, y)$ as defined by relation (2) above.

SOLUTION. $w = z^2 = (x + yi)^2 = x^2 + 2xyi - y^2$
$$= x^2 - y^2 + i(2xy).$$

Hence, $$u(x, y) = x^2 - y^2,$$

and $$v(x, y) = 2xy.$$

We will now consider some of the distinctions between the functions of a real variable and those of a complex variable. In Sec. 3.9 we discussed the graphical representation of the functional relation $y = f(x)$ by plotting points in the rectangular coordinate system, using the X-axis for *real* values of the variable x and the Y-axis for *real* values of the variable y. But the situation is entirely different for the graphical representation of the functional relation $w = f(z)$ as given by equation (1) above. Here, in order to plot the independent variable $z = x + yi$, we require the entire xy-plane or z-plane (Sec. 8.4), and no room is left for the corresponding values of the function w. To meet this situation, we create another coordinate plane called the uv-plane or w-plane, in accordance with relation (2). That is, just as we plotted the point $z = x + yi$ as a point with the real coordinates (x, y)

in the z-plane, so we plot the corresponding point $w = u + vi$ as a point with the real coordinates (u, v) in the w-plane. We thus study the geometric or graphical representation of the functional relation $w = f(z)$ by a correspondence between points of the z- and w-planes. With suitable definitions and restrictions, this correspondence is known as *conformal representation*. It is of considerable importance in the theory and application of analytic functions of a complex variable.

The student who has studied logarithms will recall that in the relation $y = \log x$, the number x is restricted to positive values, that is, logarithms of positive numbers only are considered. But in the theory of functions of a complex variable, it is shown that logarithms of negative numbers also exist; in fact, it is shown that there are logarithms of any of the numbers of our algebra, real or complex.

We have another distinction in trigonometry. In elementary trigonometry, the various trigonometric functions are restricted to real values of the angle. Thus, in the relation $y = \sin x$, the angle x is permitted to take on only real values and y can never have an absolute value greater than unity. From this point of view, the angle x has no meaning in the relation $\sin x = 2$. But if x is permitted to assume all values, real or complex, this last relation has a definite meaning in the field of functions of a complex variable.

There are many other distinctions between functions of real and complex variables, but the few examples given above illustrate the fact that the theory of functions of a complex variable has served to unite many concepts previously considered disconnected. Permitting the independent variable to assume any values, real or complex, is a procedure appropriately called *generalization*, for it gives more general results whose existence would otherwise be unknown. The theory of functions of a complex variable is an important subject of advanced study and is the basis of analysis with its many applications to mathematical physics, particularly in the fields of hydrodynamics, heat, and electricity.

EXERCISES. GROUP 29

1. Show that the set of all positive and negative integers and zero constitutes an infinite group with respect to the operation of subtraction.

2. Show that the set of Ex. 1 does not constitute a group with respect to multiplication.

3. Show that the set of all real numbers constitutes a group with respect to addition but not to multiplication.

4. Show that the set of all positive rational numbers constitutes a group with respect to multiplication.

5. Show that the set of all rational numbers constitutes a group with respect to addition.

6. Postulate 1 for a group states that any two elements which are combined need not necessarily be distinct. If the rule of combination is multiplication, this implies that the square of any element must also be an element of the group. Verify this fact for the group of the Example of Sec. 8.7 by showing that the square of every element is also an element of the group.

7. Show that the four fourth roots of unity constitute a group with respect to multiplication.

8. Show that unity and all integral powers (positive and negative) of the imaginary unit i form a group with respect to multiplication.

9. Prove that the identity element e of a group is unique. *Hint:* Assume there are two identity elements and then show that they are identical.

10. Prove that for every element a of a group, its inverse a' is unique. *Hint:* Assume a has two inverses and then show that they are identical.

In Exs. 11–16, $1, \omega, \omega^2$ represent the three cube roots of unity.

11. Show that $1 + \omega + \omega^2 = 0$.

12. Using the result of Ex. 11, show that $(1 + \omega)^3 = -1$.

13. Using the result of Ex. 11, show that $(1 + \omega^2)^4 = \omega$.

14. Show that $1 + 1/\omega + 1/\omega^2 = 0$.

15. Show that $1 + \omega + 1/\omega = 0$.

16. If N is any real number, show that the three cube roots of N are $\sqrt[3]{N}$, $\sqrt[3]{N}\omega$, and $\sqrt[3]{N}\omega^2$, where $\sqrt[3]{N}$ is the principal cube root of N.

17. From Theorem 4 (Sec. 8.6), the formula for the n nth roots of unity is given by

$$\cos\frac{k \cdot 360^\circ}{n} + i \sin\frac{k \cdot 360^\circ}{n}, \quad k = 0, 1, 2, \cdots, n-1.$$

Represent the nth root for $k = 1$ by ω, that is, let

$$\omega = \cos\frac{360^\circ}{n} + i \sin\frac{360^\circ}{n}.$$

Then show that for $k = 2, 3, 4, \cdots, n-1$, the successive nth roots of unity are given, respectively, by $\omega^2, \omega^3, \omega^4, \cdots, \omega^{n-1}$. Hence show that the n nth roots of unity are given by $1, \omega, \omega^2, \cdots, \omega^{n-1}$.

18. Using the result of Ex. 17, show that the product of any two nth roots of unity is also one of the nth roots of unity.

19. Using the result of Ex. 17, show that (a) the square of each nth root of unity is also one of the nth roots of unity, (b) the reciprocal of each nth root of unity is also one of the nth roots of unity.

20. Using the results of Exs. 18 and 19, generalize the Example of Sec. 8.7, that is, show that the n nth roots of unity constitute a group with respect to the operation of multiplication.

21. Show that any vector is equal to the sum of its horizontal and vertical components.

22. If the vector b is the negative of the vector a, show that the horizontal and vertical components of b are, respectively, the negatives of the horizontal and vertical components of a.

23. Show how the method of finding the sum of two vectors graphically may be extended to the sum of three or more vectors.

24. Draw a figure illustrating the graphical method of obtaining the difference of two given vectors.

25. Show that the sum of any number of given vectors is another vector whose horizontal component is equal to the algebraic sum of the horizontal components of the given vectors and whose vertical component is equal to the algebraic sum of the vertical components of the given vectors.

26. Find the sum of the two vectors $a(3\sqrt{2}, 45°)$ and $b(2, 120°)$ both graphically and analytically.

27. Find the difference $a - b$ of the two vectors a and b of Ex. 26, both graphically and analytically.

In each of Exs. 28–30, for the given function of $z = x + yi$, find the functions $u(x, y)$ and $v(x, y)$ where $w = u(x, y) + iv(x, y)$.

28. $w = z^2 + 2z - 1$. 29. $w = 1/z$, $z \neq 0$. 30. $w = z^3$.

9

Variation

9.1. INTRODUCTION

In a functional relation, say $y = f(x)$, we have seen that a change in the variable x is generally accompanied by a change in the variable y, and vice versa. We then say that y varies as x or that x varies as y, and refer to this correspondence as *functional variation.*

There are a great many varieties of functional variation. Here we shall study several specific types which we may appropriately include under the title of *special variation.* These types are special in the sense that they follow a definite law or relation which, in general, may be readily stated in words and expressed in the form of an equation. Such types occur frequently in geometry and physics. For example, the variation in the area of a triangle bears a fixed relation to any variations in the lengths of its base and altitude. The next section lays the ground work for the solution of problems involving special variation.

9.2. DEFINITIONS AND PROPERTIES

The *ratio* of one number y to another nonzero number x is *defined* as the quotient y/x. It is important to observe that a ratio is a pure number and as such is the quotient of like quantities. Thus the ratio of 2 ft to 3 yd is 2/9.

The variable y is said to *vary directly* as the variable x if their ratio is always a constant, that is, if $y/x = k$ or

(1) $$y = kx,$$

where the constant k is called the *constant of variation* or the *constant of proportionality*.

Thus the circumference C of a circle varies directly as the radius r since $C = 2\pi r$ where 2π is the constant of variation.

The variable y is said to *vary inversely* as the variable x if y varies directly as the reciprocal of x. We then write

$$(2) \qquad y = \frac{k}{x},$$

where k is the constant of variation.

Thus, the intensity of illumination I upon a given surface varies inversely as the square of the distance d between the surface and the source of light, that is, $I = k/d^2$, where k is the constant of variation.

The two cases of special variation just considered each involve only two variables. But we may also have variation involving more than two variables. One variable is said to *vary jointly* as two or more other variables if it varies directly as their product. Thus, w varies jointly as x, y, and z if $w = kxyz$, where k is the constant of variation. Furthermore, if w varies jointly as x, y, and $1/z$ so that we may write $w = \dfrac{kxy}{z}$, we also say that w varies directly as x, directly as y, and inversely as z. This last type is sometimes called *combined variation* but it is evidently a particular example of joint variation.

An important relation in joint variation is given by

Theorem 1. *If z varies directly as x when y is constant, and if z varies directly as y when x is constant, then z varies jointly as x and y.*

PROOF. From the first part of the hypothesis,

$$(3) \qquad z = k_1x, \quad y \text{ constant.}$$

When z changes to some other value, say $\bar{z}$, let x change correspondingly to x' so that

$$(4) \qquad \bar{z} = k_1x'.$$

Dividing (3) by (4), member by member, we have

$$(5) \qquad \frac{z}{\bar{z}} = \frac{x}{x'}.$$

From the second part of the hypothesis,

$$(6) \qquad z = k_2y, \quad x \text{ constant.}$$

Now, while x retains its value x' (hereafter a constant), let y change to y' (a constant). thus causing z to change from $\bar{z}$ to z' (a constant), so that from (6) we have

$$(7) \qquad \frac{\bar{z}}{z'} = \frac{y}{y'}.$$

Multiplying (5) by (7), member by member, we have

$$\frac{z}{z'} = \frac{xy}{x'y'}$$

or

$$z = \frac{z'}{x'y'}\,xy.$$

Replacing $\frac{z'}{x'y'}$ by k, a constant, we have the required relation

$$z = kxy.$$

This completes the proof.

Corollary. *If z varies directly as x when y is constant, and if z varies inversely as y when x is constant, then z varies jointly as x and $1/y$, that is,* $z = k\frac{x}{y}$.

A simple geometric example of Theorem 1 is the relation between the area of a triangle and its base and altitude. Thus the area varies directly as the base when the altitude is constant, and varies directly as the altitude when the base is constant, and consequently varies jointly as the base and altitude.

An important physical example of the corollary to Theorem 1 is the relation existing among the volume V, the pressure P, and the absolute temperature T of a given mass of a perfect gas. Here V varies directly as T when P is constant, and V varies inversely as P when T is constant. Hence, in accordance with the corollary, $V = R\frac{T}{P}$ or, as it is usually written, $PV = RT$, where R is the constant of variation. This relation is sometimes called the *characteristic equation* of a gas.

9.3. PROBLEMS IN VARIATION

Next we consider a variety of problems in special variation. In the solution of such problems, we first write the given law of variation in the form of an equation containing the constant of variation k. If we then determine the value of k from the given data, we have a relation from which we may compute a required quantity. The process is illustrated in the following two examples.

Example 1. w varies jointly as x and the square of y and inversely as the cube of z. If $w = 8$ when $x = 2$, $y = 6$, and $z = 3$, find w when $x = 5$, $y = 4$, and $z = 2$.

SOLUTION. We first write the given type of variation in the form of the equation

$$(1) \qquad w = \frac{kxy^2}{z^3},$$

where k is the constant of variation. Substituting the given set of values for w, x, y, and z in this equation, we have

$$8 = \frac{k \cdot 2 \cdot 6^2}{3^3},$$

whence $k = 3$ and relation (1) may be written as

$$w = \frac{3xy^2}{z^3}.$$

Hence, for $x = 5$, $y = 4$, and $z = 2$, we have

$$w = \frac{3 \cdot 5 \cdot 4^2}{2^3} = 30.$$

Example 2. The pressure P of the wind on a vertical plane surface varies jointly as the area A of the surface and the square of the wind velocity v. A wind velocity of 20 mi per hour produces a pressure of 10 lb on a square foot. Find the wind velocity that will produce a pressure of 360 lb on a square yard.

SOLUTION. The law of variation is

$$(2) \qquad P = kAv^2.$$

Substituting the given data in (2), we have

$$10 = k \cdot 1 \cdot \overline{20}^2,$$

whence $k = 1/40$ and relation (2) may be written

$$P = \frac{1}{40} Av^2.$$

In computing v from this last relation we must be careful to use the proper units, for the constant of variation k was obtained on the basis that v is in miles per hour, P is in pounds, and A is in square feet. Hence the area of 1 sq yd must be changed to 9 sq ft. Then for our required velocity v, we have

$$360 = \frac{1}{40} \cdot 9 \cdot v^2,$$

whence $v^2 = \dfrac{360 \cdot 40}{9} = 40 \cdot 40$ and $v = 40$ mi per hour.

For problems in variation it is generally desirable to obtain the constant of variation since we then have a complete formula for computations. However, in some cases the constant is not required or may be unobtainable. For example, we may be interested only in the effect on one variable due to changes in the other variable(s) as illustrated in

Example 3. The electrical resistance R of a wire of circular cross section varies directly as the length L and inversely as the square of the diameter d of the wire. Find the per cent change in the resistance of a given wire if the length is increased by 40 per cent and the diameter by 30 per cent.

SOLUTION. The law of variation is expressed by

$$R = \frac{kL}{d^2}, \tag{3}$$

where the constant of variation k depends upon the nature of the material of the wire.

We are now interested in a new value of R, say R_1, which is due to changing L to $1.4L$ and d to $1.3d$. This new set of values gives us the relation

$$R_1 = \frac{k(1.4L)}{(1.3d)^2}, \tag{4}$$

where k has the same value as in (3).

From (3) and (4) we have

$$\frac{R_1}{R} = \frac{1.4Ld^2}{1.69d^2L} = \frac{1.4}{1.69} = 0.828,$$

whence $R_1 = 0.828R$, that is, the resistance is decreased by 17.2 per cent, whatever may be the material of the wire.

EXERCISES. GROUP 30

1. If y varies directly as x and $y = 8$ when $x = 4$, find y when $x = 7$.
2. If y varies inversely as x and $y = 3$ when $x = 5$, find x when $y = 5$.
3. If z varies directly as x and inversely as y, and if $z = 2$ when $x = 3$ and $y = 9$, find z when $x = -10$ and $y = 12$.
4. If y varies inversely as $x^2 + 1$ and $y = 2$ when $x = 2$, find y when $x = \pm 3$.
5. If w varies jointly as x and y and inversely as the square of z, and if $w = -20$ when $x = 6, y = 5$, and $z = 3$, find y when $x = 8, z = 2$, and $w = 24$.
6. If z varies directly as $(x - y)/(x + y)$ and $z = 2$ when $x = 7$ and $y = 5$, find x when $y = 3$ and $z = 6$.

7. y varies directly as the sum of two quantities, the first of which varies directly as x and the second inversely as x. If $y = 3$ when $x = 2$, and $y = 7$ when $x = 3$, find the functional relation between x and y.

8. y varies directly as the difference of two quantities, the first of which (the minuend) varies inversely as x and the second inversely as x^2. If $y = 12$ when $x = 1$, and $y = 4$ when $x = 2$, find the functional relation between x and y.

9. y varies directly as the sum of three quantities, the first of which varies directly as x^3, the second directly as x^2, and the third directly as x. If $y = 4$ when $x = 1$, $y = 14$ when $x = 2$, and $y = -10$ when $x = -1$, find the functional relation between x and y.

In each of Exs. 10–15, prove the stated theorem.

10. If x varies directly as y and y varies directly as z, then x varies directly as z.

11. If x varies inversely as y and y varies inversely as z, then x varies directly as z.

12. If x varies directly as z and y varies directly as z, then $x \pm y$ varies directly as z.

13. If x varies directly as z and y varies directly as z, then $\sqrt{xy}$ varies directly as z.

14. If x varies directly as y, then x^n varies directly as y^n.

15. If x varies directly as y and u varies directly as v, then xu varies directly as yv.

16. Establish the corollary to Theorem 1 of Sec. 9.2.

17. Generalize Theorem 1 of Sec. 9.2 by showing that if z varies directly in turn as each one of the variables $x_1, x_2, \cdots, x_n$ when all the remaining variables are constant, then z varies jointly as $x_1, x_2, \cdots, x_n$.

18. The distance traversed by a body starting from rest and falling freely in a vacuum varies directly as the square of the time of descent. If a body has fallen 16 ft at the end of 1 second, find the distance fallen at the end of 4 seconds.

19. For a body falling as specified in Ex. 18, the velocity acquired varies directly as the time of descent. If a body acquires a velocity of 64 ft per second at the end of 2 seconds, find the time required for the body to acquire a velocity of 160 ft per second.

20. Boyle's law states that at a constant temperature, the volume of a given mass of gas varies inversely as the pressure to which it is subjected. If a given mass of gas has a volume of 140 cu in. under a pressure of 20 lb, find its volume when the pressure is 35 lb.

21. The time of vibration of a simple pendulum varies directly as the square root of its length. If the time of vibration of a pendulum 10 in. long is 1 second, find the time of vibration of a pendulum 40 in. long.

22. The volume of a given mass of gas varies directly as the absolute temperature and inversely as the pressure. If the volume is 10 cu ft when the temperature is 300° and the pressure is 12 lb per sq in., find the volume when the temperature is 320° and the pressure is 16 lb per sq in.

23. The safe load S of a horizontal beam of rectangular cross section when supported at both ends varies jointly as the breadth b and the square of the depth d and inversely as the length L between supports. A 2 by 4 in. beam 6 ft long and resting on the 2 in. side ($b = 2$ in.) will safely support 800 lb. Find the safe load for the same beam when it rests on its 4 in. side ($b = 4$ in.).

24. Ohm's law states that the current flowing through a conductor varies directly as the electromotive force and inversely as the resistance. If the resistance is decreased by 10 per cent, find the per cent change in the electromotive force required to increase the current by 20 per cent.

25. The area of the lateral surface of a right circular cylinder varies jointly as the radius of its base and its altitude. If the radius is increased by 20 per cent, find the per cent change in the altitude so that the area of the lateral surface remains unchanged.

26. The lift of an airplane wing varies jointly as the area of the wing and the square of the plane's velocity. Determine the per cent change in the lift if the wing area is decreased 25 per cent and the velocity is increased 25 per cent.

27. The volume of a right circular cone varies jointly as the square of the radius of its base and its altitude. If the radius is increased by 10 per cent, find the per cent change in the altitude so that the volume remains unchanged.

28. The illumination on a screen varies directly as the intensity of the source of light and inversely as the square of the distance from the source. Find the per cent change in the illumination if the intensity of the source is increased by 20 per cent and the distance from the source is increased by 10 per cent.

29. The frequency of vibration of a stretched string varies directly as the square root of the tension on the string and inversely as the product of the length and diameter of the string. Find the per cent change in the frequency if the tension is increased by 20 per cent, the length is increased by 15 per cent, and the diameter is decreased by 10 per cent.

30. Newton's law of gravitation states that the force of attraction between two bodies varies directly as the product of their masses and inversely as the square of the distance between them. If one mass is increased by 10 per cent and the distance between the masses is decreased by 10 per cent, find the per cent change in the other mass if the force is to be increased by 10 per cent.

9.4. VARIATION OF ALGEBRAIC FUNCTIONS

We will now consider a more general type of variation in algebraic functions. For example, in the relation $x^2 + xy + y^2 = 4$, defining y as an implicit function of x, it is impossible to express the law of variation between the variables x and y by a comparatively simple statement, as in the problems of special variation discussed in the preceding sections. We may, however, obtain an excellent idea of the variation between x and y in this case from the graphical representation of the functional relation. We have previously discussed the graphical representation of functions of a single variable, and the student will find it helpful to reread Sec. 3.9 at this point. We shall now continue our study of graphical representation by making a closer examination of a functional relation involving two variables, x and y, *before* attempting to plot it. This is termed a *discussion* of the equation of the graph and has several advantages. It often serves to reduce the amount of labor involved in computing the coordinates of points on the graph. It may also help us to avoid serious errors in the appearance of a graph *between* plotted points. Several items in such a discussion will now be described.

The first item we shall consider in connection with the discussion of an equation is the *intercepts*, if any, of the graph on the coordinate axes. The *intercept* of a graph on the X-axis is the abscissa of the point of intersection of the graph and the X-axis. The *intercept* of a graph on the Y-axis is the ordinate of the point of intersection of the graph and the Y-axis. The method of obtaining the intercepts is quite obvious from the definitions. Since the intercept on the X-axis is the abscissa of a point lying on the X-axis, the ordinate of that point is zero. Hence, by setting $y = 0$ in the equation of the graph, the solution of the resulting equation for *real* values of x will give the intercepts on the X-axis. Similarly, by setting $x = 0$ in the equation of the graph, the solution of the resulting equation for *real* values of y will give the intercepts on the Y-axis. It is important to observe that the intercepts on the X-axis represent the values of *real zeros* (Sec. 3.9).

Another item of great importance in the discussion of an equation is the *extent* of its locus or graph. By this term we mean a determination of the range of the real values which x and y may assume in the equation of the locus. This information is useful in two respects: (1) It gives the general location of the graph in the coordinate plane. (2) It indicates whether the locus is a closed curve or is indefinite in extent. As we shall soon see, the range of the real values of x and y is determined simply by solving the given equation for y in terms of x and also for x in terms of y.

The two items just described are illustrated in

Example 1. Discuss the equation $x^2 + xy + y^2 = 4$ and plot its graph.

SOLUTION. From the given equation, for $y = 0$, $x^2 = 4$ whence $x = \pm 2$, the intercepts on the X-axis. Similarly, by setting $x = 0$ in the given equation, we find $y = \pm 2$, the intercepts on the Y-axis.

Next, in order to determine the extent of the graph we solve the given equation for y in terms of x and also for x in terms of y. First we write the given equation in the form

$$y^2 + xy + x^2 - 4 = 0$$

and consider it as a quadratic equation in the variable y, the variable x now being treated as a constant. Then by the quadratic formula we have

$$y = \frac{-x \pm \sqrt{x^2 - 4x^2 + 16}}{2}, \quad \text{or}$$

$$y = \frac{-x \pm \sqrt{16 - 3x^2}}{2}. \tag{1}$$

Now, since we are dealing only with *real* values of x and y, it follows from (1) that we must have $16 - 3x^2 \geq 0$. By the methods of Chapter 6 (Sec. 6.5), we find that this relation holds when x is restricted to the range of values given by $-\frac{4}{3}\sqrt{3} \leq x \leq \frac{4}{3}\sqrt{3}$.

Similarly, by writing the given equation in the form

$$x^2 + yx + y^2 - 4 = 0$$

and considering it as a quadratic equation in the variable x, treating y as a constant, we find by the quadratic formula that

$$x = \frac{-y \pm \sqrt{16 - 3y^2}}{2}.$$

Hence, for real values, y is restricted to the range given by $-\frac{4}{3}\sqrt{3} \leq y \leq \frac{4}{3}\sqrt{3}$.

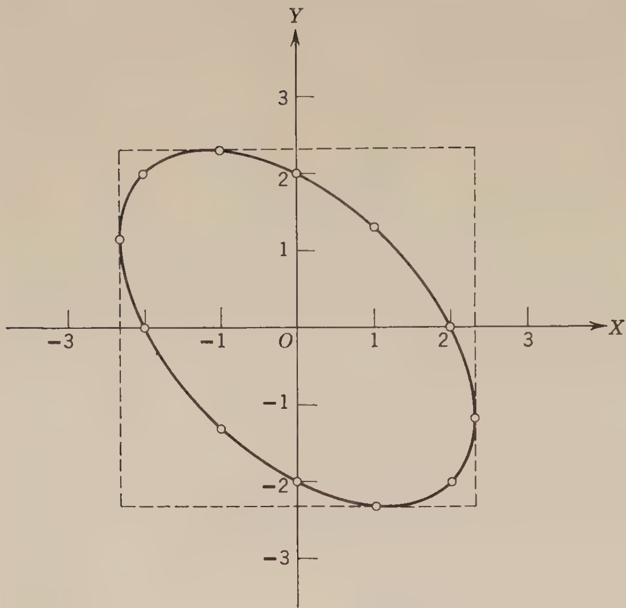
It follows, therefore, that all points on the graph must be entirely within (or on) a square whose center is at the origin and whose sides are parallel to the coordinate axes and $\frac{8}{3}\sqrt{3}$ units apart. This means, of course, that the graph is a *closed curve*, as shown in Fig. 35.

In addition to the intercepts, we may obtain the coordinates of additional points on the graph from relation (1). These values are shown in the table accompanying the graph (an ellipse).

Some curves have associated with them one or more lines called *asymptotes* which are very useful in constructing the graph. For such lines we have the

Definition. If, for a given curve, there exists a line such that, as a point on the curve recedes indefinitely far from the origin, the distance from that point to the line continually decreases and approaches zero, then the line is called an *asymptote* of the curve.

In our work we shall consider only horizontal and vertical asymptotes, that is, asymptotes which are parallel to the X- and Y-axes, respectively. Many curves have no asymptotes, but if a curve does have any horizontal or vertical asymptotes, they may be found by means of the solved forms



x	y
0	± 2
1	$\dfrac{-1 \pm \sqrt{13}}{2}$
-1	$\dfrac{1 \pm \sqrt{13}}{2}$
2	$0, -2$
-2	$0, 2$
$\frac{4}{3}\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$
$-\frac{4}{3}\sqrt{3}$	$\frac{2}{3}\sqrt{3}$

Figure 35

used in determining the extent of the curve as we shall see in the next example.

Example 2. Discuss the equation $x^2y - x^2 - y = 0$ and plot its graph.

SOLUTION. It is readily seen that the only intercepts on the axes are given by the origin.

The solution of the given equation for y in terms of x is

$$y = \frac{x^2}{x^2 - 1}. \tag{2}$$

In (2), y is defined for all values of x except ± 1. Hence the graph is not a closed curve but is indefinite in extent. For $x > 1$ and $x < -1$, y is positive; for values of x in the range $-1 < x < 1$, y is negative or zero. As x approaches $+1$ or -1, y increases numerically without limit so that, in accordance with our definition above, the lines $x = 1$ and $x = -1$ represent vertical asymptotes.

The solution of the given equation for x in terms of y is

$$x = \pm\sqrt{\frac{y}{y - 1}}. \tag{3}$$

In (3), x is not defined for $y = 1$. Also, x is complex for values of y in the range $0 < y < 1$, and thus such values of y must be excluded. As y approaches 1 through values greater than 1, x increases numerically without limit; hence the line $y = 1$ represents a horizontal asymptote.

The conclusions drawn from equations (2) and (3) as to the permissible ranges of values of x and y give us a good idea of the location of the graph in the coordinate plane. There are three definite regions in which the

x	y
0	0
$\pm\frac{1}{4}$	$-\frac{1}{15}$
$\pm\frac{1}{2}$	$-\frac{1}{3}$
$\pm\frac{3}{4}$	$-\frac{9}{7}$
$\pm\frac{5}{4}$	$\frac{25}{9}$
$\pm\frac{3}{2}$	$\frac{9}{5}$
$\pm\frac{7}{4}$	$\frac{49}{33}$
± 2	$\frac{4}{3}$
$\pm\frac{9}{4}$	$\frac{81}{65}$

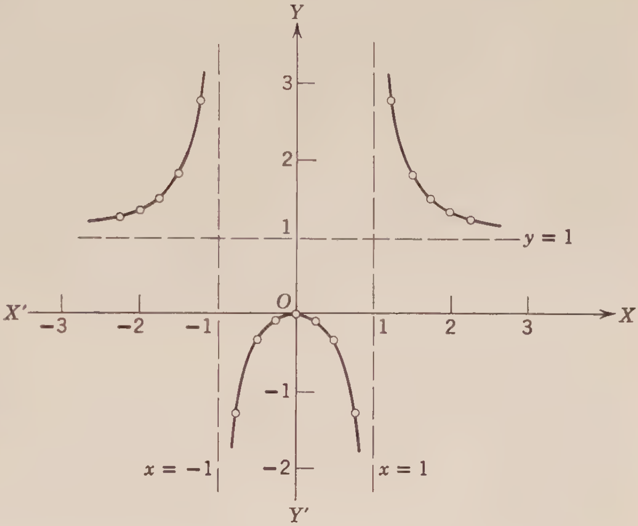


Figure 36

locus appears: above the line $y = 1$ and to the right of the line $x = 1$; above the line $y = 1$ and to the left of the line $x = -1$; and below the X-axis and between the lines $x = 1$ and $x = -1$. The locus is evidently open.

The coordinates of points may be obtained from (2) within the limits set above. Several such pairs of values are given in the table in Fig. 36. Asymptotes are shown by dotted lines in the graph of the same figure. The student should note the advantage of using the asymptotes of a curve, when they exist, in plotting the locus. Asymptotes act as *guiding lines* in the graph.

Summarizing, we list the following five distinct steps in discussing the equation of a curve and plotting its graph:

1. Determination of any intercepts on the coordinate axes.
2. Determination of the extent of the graph.

3. Determination of any vertical and horizontal asymptotes the curve may possess.
4. The computation of the coordinates of a sufficient number of points to obtain an adequate graph.
5. The actual plotting of the locus.

The student should always make it a particular point to see that the graph and discussion of an equation are in complete agreement.

NOTE. The above discussion of an equation is far from exhaustive, but it is sufficient for our present purposes and the scope of this book. A more detailed examination and analysis of an equation are given in analytic geometry and calculus and include the determination of symmetry, maximum and minimum points, points of inflection, and other points which may be significant on the graph.

We may note also that there are situations in which a set of corresponding observations may be available but not an equation. A graph of such observations is often of great value in exhibiting significant data and in drawing important conclusions. This occurs frequently in experimental work, statistics, and curve fitting.

EXERCISES. GROUP 31

In each of the following exercises, discuss the given equation and plot its graph.

1. $x^2 + y^2 = 4$.
2. $9x^2 + 4y^2 = 36$.
3. $4x^2 + 9y^2 = 36$.
4. $9x^2 - 4y^2 = 36$.
5. $4y^2 - 9x^2 = 36$.
6. $x^2 + y^2 - 4y = 0$.
7. $x^2 - 6x + y^2 = 0$.
8. $4x^2 - y^2 - 2y - 2 = 0$.
9. $8x^3 - y = 0$.
10. $y = x^3 + x^2 - 9x - 9$.
11. $x^3 - x - y = 0$.
12. $x^2 - y^3 = 0$.
13. $x - y^4 + 9y^2 = 0$.
14. $xy - y - 1 = 0$.
15. $x^3 + xy^2 - y^2 = 0$.
16. $x^2 + xy + y^2 = 1$.
17. $x^2 - xy + y^2 = 1$.
18. $x^2 + 2xy - y^2 = 2$.
19. $x^2 - xy - 3y^2 = 1$.
20. $x^3 + y^2 - 4y + 4 = 0$.
21. $x^2y - 4y - x = 0$.
22. $xy^2 - 9x - y - 1 = 0$.
23. $x^2y - xy - 2y - 1 = 0$.
24. $x^2 - xy + 5y = 0$.
25. $x^2y^2 - 4x^2 - 4y^2 = 0$.
26. $y^2 = x(x - 1)^2$.
27. $y^2 = (x - 1)(x - 2)$.
28. $y^2 = (x - 1)(x - 2)(x - 3)$.
29. $y^2 = x(x + 1)(x + 2)$.
30. $y^2 = x(x + 2)^2$.

10

Progressions

10.1. INTRODUCTION

In this chapter we shall study the properties of certain special sets of quantities. They are special in the sense that each member or term of the set is formed *in order* according to some common law. For example, in the set of n numbers,

(1) $$3, 5, 7, \cdots, 2n + 1,$$

the successive terms of the set are formed in order by multiplying the number of the term by 2 and increasing the result by 1. Thus the first term is $2(1) + 1 = 3$, the second term is $2(2) + 1 = 5$, the third term is $2(3) + 1 = 7$, and so on.

Sets of this type are so important that we have a special name for them, as given in the

Definition. A *sequence* of numbers is an *ordered set* of numbers formed in accordance with a given law.

The essential requirement for a sequence is that there must be a law or formula whereby it is possible to obtain any member of the sequence. Thus, if u_n represents the nth term of a sequence, then we must have an expression for u_n in terms of n, that is, a function of n. Thus, in the example given above, $u_n = 2n + 1$, a relation which permits us to obtain any desired term of the sequence.

If a sequence has a last term, it is called a *finite sequence;* if it has no last term, that is, if the number of terms is unlimited, it is called an *infinite sequence.*

NOTE. The indicated sum of the terms of a sequence is called a *series*; such a series is said to be *finite* or *infinite* depending on whether the sequence is finite or infinite. Infinite series are the subject of special study in the calculus; they are also of fundamental importance in the theory of functions.

In the following sections we shall study three different types of finite sequences and one type of infinite sequence.

10.2. ARITHMETIC PROGRESSION

We first lay down the following

Definition. An *arithmetic progression* is a sequence of numbers such that each term after the first is obtained by adding to the preceding term a fixed number called the *common difference*.

An example of an arithmetic progression is the sequence (1) of Sec. 10.1.

In accordance with the definition, an arithmetic progression may be written in the form

$$a_1, \quad a_1 + d, \quad a_1 + 2d, \quad a_1 + 3d, \cdots, \tag{1}$$

where a_1 is called the *first term* and d is the common difference.

If a_n represents the nth term of the sequence (1), then

the second term is $a_2 = a_1 + d,$

the third term is $a_3 = a_1 + 2d,$

the fourth term is $a_4 = a_1 + 3d,$

and, in general, the nth term is

$$a_n = a_1 + (n - 1)d. \tag{2}$$

We will now obtain an expression for s_n, the sum of the first n terms of the sequence (1), which may be written

$$s_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_n - 2d) + (a_n - d) + a_n. \tag{3}$$

Writing the terms of the right member of (3) in reverse order, we have

$$s_n = a_n + (a_n - d) + (a_n - 2d) + \cdots + (a_1 + 2d) + (a_1 + d) + a_1. \tag{4}$$

Adding (3) and (4), member by member, we have

$$2s_n = (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) = n(a_1 + a_n)$$

whence

$$s_n = \frac{n}{2}(a_1 + a_n). \tag{5}$$

We record these results in

Theorem 1. *If a_1 is the first term, a_n the nth term, d the common difference and s_n the sum of the first n terms of an arithmetic progression, then we have the two independent relations:*

$$a_n = a_1 + (n - 1)d,$$

and

$$s_n = \frac{n}{2}(a_1 + a_n).$$

From the two relations of Theorem 1 we also have the following useful expression for s_n in place of relation (5):

$$s_n = \frac{n}{2}[2a_1 + (n - 1)d]. \tag{6}$$

Theorem 1 may also be established by mathematical induction (Sec. 7.2).

It is important to observe that the five elements, a_1, a_n, d, n, and s_n, of an arithmetic progression are connected by two independent relations. Hence, if any three of these elements are known, the other two may be determined.

Example 1. For the arithmetic progression $3, 5, 7, 9, \cdots$, find the twelfth term and the sum of the first twelve terms.

SOLUTION. For this progression, $a_1 = 3$, $d = 2$, $n = 12$. Hence by Theorem 1,

$$a_{12} = a_1 + (n - 1)d = 3 + 11 \cdot 2 = 25,$$

and

$$s_{12} = \frac{n}{2}(a_1 + a_n) = \frac{12}{2}(3 + 25) = 168.$$

Example 2. For the arithmetic progression in which $a_1 = 2$ and $d = 3$, find how many terms must be taken so that the sum may be 155.

SOLUTION. In this problem we are given $a_1 = 2$, $d = 3$, and $s_n = 155$, and are required to find n. Since we do not know a_n, it will be convenient to use equation (6) above for s_n, whence

$$155 = \frac{n}{2}[2 \cdot 2 + (n - 1)3],$$

$$310 = 4n + 3n^2 - 3n,$$

$$3n^2 + n - 310 = 0.$$

Factoring,

$$(3n + 31)(n - 10) = 0,$$

whence

$$n = -\frac{31}{3}, \quad 10.$$

Since n must be a positive integer, the required number of terms is 10.

In an arithmetic progression, the terms between any two given terms a and b are called the *arithmetic means* between a and b. The terms a and b are then called the *extremes*. Thus, in the arithmetic progression 3, 6, 9, 12, 15, 18, · · ·, the arithmetic means between 6 and 18 (extremes) are 9, 12, 15. The determination of the arithmetic means between any two given numbers is illustrated in

Example 3. Insert five arithmetic means between 9 and -3.

SOLUTION. We are required to find five numbers which, with 9 and -3 as extremes, will form an arithmetic progression. Hence we need only find the common difference d for an arithmetic progression of 7 terms with $a_1 = 9$ and $a_7 = -3$. Substituting in the relation

$$a_n = a_1 + (n - 1)d,$$

we have

$$-3 = 9 + 6d,$$

whence

$$d = -2.$$

Thus the five arithmetic means between 9 and -3 are $9 - 2 = 7$, 5, 3, 1, -1. As a check on our work we note that by adding $d = -2$ to the last arithmetic mean -1, we obtain -3, the next term and extreme.

The single arithmetic mean between two given numbers is called their *arithmetic mean*. Let A be the arithmetic mean of two given numbers a and b, so that a, A, b are in arithmetic progression. Then, for their common difference, we have

$$d = A - a = b - A,$$

whence

$$2A = a + b,$$

and

$$A = \frac{a + b}{2},$$

that is, *the arithmetic mean of two given numbers is equal to one half their sum*. It is also frequently called their *average*.

EXERCISES. GROUP 32

In each of Exs. 1–6, for the given arithmetic progression, find a_n and s_n for the indicated number of terms.

1. 2, 6, 10, · · · to 11 terms.
2. -3, -1, 1, · · · to 9 terms.
3. 9, 7, 5, · · · to 14 terms.
4. 10, 9, 8, · · · to 20 terms.
5. -8, $-\frac{13}{2}$, -5, · · · to 16 terms.
6. 3, $\frac{8}{3}$, $\frac{7}{3}$, · · · to 24 terms.

In each of Exs. 7–14, three of the five elements of an arithmetic progression are given. Find the other two elements.

7. $a_1 = 5$, $d = -3$, $n = 8$.
8. $a_1 = -3$, $a_n = 8$, $s_n = 30$.

9. $a_1 = 11, d = -2, s_n = -28$.
10. $a_n = 29, s_n = 225, d = 2$.
11. $a_1 = 30, a_n = -10, s_n = 90$.
12. $n = 11, d = 2, s_n = -44$.
13. $a_1 = 45, d = -3, s_n = 357$.
14. $a_n = 9, d = 3, s_n = -66$.
15. Find the sum of all positive multiples of 3 which are less than 20.
16. Find the sum of all positive multiples of 5 which are less than 100.
17. Find the arithmetic mean of 7 and -11.
18. The arithmetic mean of two numbers is 6. If one number is 21, find the other number.
19. Insert five arithmetic means between -4 and 8.
20. Insert seven arithmetic means between 5 and 1.
21. Insert five arithmetic means between -12 and 4.
22. Insert two arithmetic means between $1 + \sqrt{2}$ and $1 - 2\sqrt{2}$.
23. The third term of an arithmetic progression is -3 and the eighth term is 2. Find the common difference and the sixth term.
24. The fourth term of an arithmetic progression is 11 and the eleventh term is 21. Find the first term and the sum of the first fifteen terms.
25. The fifth term of an arithmetic progression is 2 and the ninth term is -10. Find the seventh term and the sum of the first twelve terms.
26. The sixth term of an arithmetic progression is -9, and the twelfth term is -33. Find the common difference and the sum of the first ten terms.
27. Establish Theorem 1 of Sec. 10.2 by mathematical induction.
28. If n arithmetic means are inserted between a and b, show that the common difference is given by $d = (b - a)/(n + 1)$.
29. Find the sum of the first n positive odd integers.
30. Find the sum of the first n positive even integers.
31. Find the sum of the first $2n$ positive integers. Check the result by combining the results of Exs. 29 and 30.
32. Find the middle term of the arithmetic progression of Ex. 1.
33. Find the middle term of the arithmetic progression of Ex. 2.
34. Find the two middle terms of the arithmetic progression of Ex. 3.
35. Find the two middle terms of the arithmetic progression of Ex. 4.
36. Find the middle term of an arithmetic progression of n terms having a_1 as its first term and d as its common difference, n being odd, and show that it is equal to s_n/n.
37. Find the two middle terms of an arithmetic progression of n terms having a_1 as its first term and d as its common difference, n being even, and show that their sum is equal to $2s_n/n$.
38. Use the results of Exs. 36 and 37 to verify the results of Exs. 32–35.
39. Find the sum of the sequence $1, -3, 5, -7, 9. -11, \cdots$ to $2n$ terms.
40. Find the sum of the sequence $1, -2, 3, -4, 5, -6, \cdots$ to $2n$ terms.

41. Show that the sum of any $2n + 1$ consecutive integers is divisible by $2n + 1$.

42. If each term of an arithmetic progression is multiplied by the same nonzero quantity, show that the resulting sequence is also an arithmetic progression.

43. A freely falling body travels approximately 16 ft the first second, and in each second thereafter 32 ft more than in the preceding second. A stone is dropped from the top of a tower and reaches the ground in 6 seconds. Find the height of the tower and the distance the stone falls during the last second.

44. The sum of three numbers in arithmetic progression is 21 and the product of the first and third number is 33. Find the numbers. *Hint*: Represent the numbers by $a - d$, a, $a + d$.

45. A number consists of four digits which are in arithmetic progression. The sum of all the digits is 16 and the sum of the last two digits is 12. Find the number.

10.3. GEOMETRIC PROGRESSION

The student will note a very close analogy between this and the preceding section. We first lay down the following

Definition. A *geometric progression* is a sequence of numbers such that each term after the first is obtained by multiplying the preceding term by a fixed nonzero number called the *common ratio*.

An example of a geometric progression is $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$.

In accordance with the definition, a geometric progression may be written in the form

$$a_1, \quad a_1r, \quad a_1r^2, \cdots, \tag{1}$$

where a_1 is called the *first term* and r is the *common ratio*.

If a_n represents the nth term of the sequence (1), then $a_2 = a_1r$, $a_3 = a_1r^2$, and, in general, the nth term is

$$a_n = a_1r^{n-1}. \tag{2}$$

We will now obtain an expression for s_n, the sum of the first n terms of the sequence (1), which may be written

$$s_n = a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-2} + a_1r^{n-1}. \tag{3}$$

Multiplying both members of (3) by r, we obtain

$$rs_n = a_1r + a_1r^2 + \cdots + a_1r^{n-2} + a_1r^{n-1} + a_1r^n. \tag{4}$$

Subtracting (4) from (3), member by member, we obtain

$$s_n - rs_n = a_1 - a_1r^n,$$

or

$$s_n(1 - r) = a_1(1 - r^n),$$

whence

$$s_n = \frac{a_1(1 - r^n)}{1 - r}, \qquad r \neq 1. \tag{5}$$

We record these results as

Theorem 2. *If a_1 is the first term, a_n the nth term, r the common ratio, and s_n the sum of the first n terms of a geometric progression, then we have the two independent relations:*

$$a_n = a_1 r^{n-1},$$

and

$$s_n = \frac{a_1(1 - r^n)}{1 - r}, \qquad r \neq 1.$$

From the first relation of Theorem 2, we have $ra_n = a_1 r^n$ which, substituted in the second relation, gives us the following useful expression in place of relation (5):

$$s_n = \frac{a_1 - ra_n}{1 - r}, \qquad r \neq 1. \tag{6}$$

Theorem 2 may also be established by mathematical induction (Sec. 7.2).

It is important to observe that the five elements, a_1, a_n, r, n, and s_n, of a geometric progression are connected by two independent relations. Hence, if any three of these elements are known, the other two may be determined.

Example 1. For the geometric progression 1, 2, 4, · · · , find the seventh term and the sum of the first seven terms.

SOLUTION. For this progression, $a_1 = 1$, $r = 2$, $n = 7$. Hence, by Theorem 2,

$$a_7 = a_1 r^{n-1} = 1 \cdot 2^6 = 64,$$

and

$$s_7 = \frac{a_1(1 - r^n)}{1 - r} = \frac{1(1 - 2^7)}{1 - 2} = 127.$$

Example 2. For the geometric progression whose first term is 4, whose last term is $30\frac{3}{8}$, and the sum of whose terms is $83\frac{1}{8}$, find the common ratio and number of terms.

SOLUTION. In this problem we are given $a_1 = 4$, $a_n = 30\frac{3}{8} = \frac{243}{8}$, and $s_n = 83\frac{1}{8} = \frac{665}{8}$, and are required to find r and n. Since r and n are both unknown, we will use relation (6) above:

$$s_n = \frac{a_1 - ra_n}{1 - r}.$$

Substituting,

$$\frac{665}{8} = \frac{4 - \frac{243}{8} r}{1 - r}.$$

Multiplying by $8(1 - r)$, $665 - 665r = 32 - 243r$,

whence $\quad -422r = -633 \quad$ and $\quad r = \dfrac{3}{2}.$

Substituting in $a_n = a_1 r^{n-1}$,

we have
$$\frac{243}{8} = 4\left(\frac{3}{2}\right)^{n-1},$$

whence
$$\frac{243}{32} = \left(\frac{3}{2}\right)^{n-1},$$

and
$$\left(\frac{3}{2}\right)^5 = \left(\frac{3}{2}\right)^{n-1}.$$

Hence $\quad n - 1 = 5$ and $n = 6$.

In a geometric progression, the terms between any two given terms a and b are called the *geometric means* between a and b. The terms a and b are then called the *extremes*. The determination of the geometric means between any two given numbers is illustrated in

Example 3. Insert five geometric means between $\frac{1}{4}$ and 16.

SOLUTION. We are to find 5 numbers which, with $\frac{1}{4}$ and 16 as extremes, will form a geometric progression. Hence all that we must determine is the ratio r for a geometric progression of 7 terms with $a_1 = \frac{1}{4}$ and $a_7 = 16$. Substituting in the relation

$$a_n = a_1 r^{n-1},$$

we have
$$16 = \tfrac{1}{4} r^6,$$

whence
$$r^6 = 64,$$

and
$$r = 2.$$

Hence the 5 geometric means are $\frac{1}{4} \cdot 2 = \frac{1}{2}$, $\frac{1}{2} \cdot 2 = 1, 2, 4, 8$. As a check on our work, we note that by multiplying the last geometric mean 8 by the common ratio 2, we obtain 16, the next term and extreme.

NOTE 1. We have seen previously (Sec. 8.6, Theorem 4) that every number (except zero) has exactly n distinct nth roots. Hence, in the preceding example, there are actually six distinct values of r and therefore six distinct sets of geometric means. However, it will be sufficient for our purposes to restrict any geometric progression so that, unless otherwise specified, the terms are both real and unique. This simplicity is obtained in the preceding example by taking only the principal root (Sec. 2.13) of 64 for the value of r.

A single geometric mean between two given numbers is called their *geometric mean*. Let G be a geometric mean of two given numbers a and b so that a, G, and b are in geometric progression. Then, for their common ratio, we have

$$r = \frac{G}{a} = \frac{b}{G}$$

whence $$G^2 = ab,$$

and $$G = \pm\sqrt{ab},$$

that is, *a geometric mean of two given numbers is numerically equal to the square root of their product.* It is also called their *mean proportional.*

NOTE 2. In Note 1 above we agreed that, unless otherwise specified, the terms of a geometric progression are to be considered both real and unique. Hence, for a geometric mean G of a and b to be real, a and b must agree in sign. Furthermore, for a unique value of G, we will agree to give G the common sign of a and b. Thus the geometric mean of 3 and 48 is 12; the geometric mean of -3 and -48 is -12. Note that in both of these cases, $r = 4$, the principal square root of 16.

EXERCISES. GROUP 33

In each of Exs. 1–6, for the given geometric progression, find a_n and s_n for the indicated number of terms.

1. $2, 4, 8, \cdots$ to 10 terms.
2. $3, 6, 12, \cdots$ to 7 terms.
3. $1, 4, 16, \cdots$ to 7 terms.
4. $3, -1, \frac{1}{3}, \cdots$ to 8 terms.
5. $48, 24, 12, \cdots$ to 6 terms.
6. $2, -\frac{2}{3}, \frac{2}{9}, \cdots$ to 7 terms.

In each of Exs. 7–12, three of the five elements of a geometric progression are given. Find the other two elements.

7. $a_1 = 1, a_n = -\frac{32}{243}, r = -\frac{2}{3}$.
8. $a_1 = 2, a_{10} = -1024, n = 10$.
9. $a_1 = 2, a_6 = 64, n = 6$.
10. $a_n = 729, r = 3, s_n = 1093$.
11. $r = 2, s_7 = 635, n = 7$.
12. $n = 6, r = -\frac{1}{4}, a_1 = 16$.
13. Insert three geometric means between 16 and $\frac{1}{16}$.
14. Insert four geometric means between $\frac{1}{9}$ and -27.
15. Insert five geometric means between $\frac{1}{8}$ and 8.
16. Insert three geometric means between 2 and 8.
17. Find the geometric mean of x^2 and y^2.
18. The geometric mean of two positive numbers is 4. If one of the numbers is four times the other, find the numbers.
19. The third term of a geometric progression is 3 and the seventh term is $\frac{3}{16}$. Find the common ratio and the first term.
20. The second term of a geometric progression is -18 and the fifth term is $\frac{16}{3}$. Find the sixth term and the sum of the first five terms.
21. The third term of a geometric progression is 9 and the sixth term is 243. Find the seventh term and the sum of the first six terms.
22. Establish Theorem 2 of Sec. 10.3 by mathematical induction.

23. The formulas for s_n, as given by relations (5) and (6) of Sec. 10.3, are not valid for $r = 1$. Obtain the formula for the sum of n terms of a geometric progression whose first term is a_1 and whose common ratio $r = 1$.

24. If each term of a geometric progression is multiplied by a nonzero constant, show that the resulting sequence is also a geometric progression.

25. Show that the alternate terms of a geometric progression form another geometric progression.

26. If each term of a geometric progression is subtracted from the following term, show that the successive differences form another geometric progression.

27. If each term of a geometric progression is raised to the same positive integral power, show that the resulting sequence is also a geometric progression.

28. Show that the reciprocals of the terms of a geometric progression also form a geometric progression.

29. Each stroke of an air pump removes one tenth of the air in a tank. At the end of eight strokes, find the fraction of the air which remains in the tank.

30. The bob of a pendulum moves through a distance of 16 in. during the first swing. In each following swing, the bob moves through $\frac{3}{4}$ of the distance of the preceding swing. Find the total distance traveled by the bob in six swings.

31. A cask contains 36 gal of pure alcohol. Six gal are drawn out and replaced by water. If this operation is performed six times, find the amount of pure alcohol left in the cask.

32. A rubber ball falling from a height of 64 ft rebounds one fourth of the preceding height after each fall. Find the total distance traversed by the ball when it strikes the ground for the sixth time.

33. The arithmetic mean of two unequal positive numbers is 5 and their geometric mean is 4. Find the numbers.

34. The sum of three numbers in arithmetic progression is 15. If these numbers are increased by 2, 1, and 3, respectively, the sums are in geometric progression. Find the numbers.

35. If three different numbers a, b, and c are in geometric progression, show that $1/(b - a)$, $1/2b$, and $1/(b - c)$ are in arithmetic progression.

10.4. HARMONIC PROGRESSION

In this section we consider a special sequence of numbers for which we have the

Definition. A *harmonic progression* is a sequence of numbers whose reciprocals form an arithmetic progression.

Thus, the sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \cdots, \frac{1}{2n}, \cdots$ is a harmonic progression since $2, 4, 6, \cdots, 2n, \cdots$ is an arithmetic progression.

It is evident from this definition that problems in harmonic progression may be readily solved by considering the corresponding arithmetic progression, as shown in the examples below. It should be noted, however, that there are no general formulas for the *n*th term and the sum of *n* terms of a harmonic progression.

Example 1. The second term of a harmonic progression is $\frac{1}{5}$ and the eighth term is $\frac{1}{23}$. Find the fifth term.

SOLUTION. We solve this problem by considering the corresponding arithmetic progression in which $a_2 = 5$ and $a_8 = 23$. Hence, by the formula of Sec. 10.2, $a_n = a_1 + (n - 1)d$, we have

$$5 = a_1 + d,$$

and

$$23 = a_1 + 7d.$$

By subtraction, $18 = 6d$, whence $d = 3$ and $a_1 = 2$. Hence, $a_5 = a_1 + 4d = 2 + 4 \cdot 3 = 14$, and the required fifth term of the harmonic progression is $\frac{1}{14}$.

In a harmonic progression, the terms between any two given terms *a* and *b* are called the *harmonic means* between *a* and *b*. The determination of the harmonic means between any two given numbers is illustrated in

Example 2. Insert four harmonic means between $\frac{1}{7}$ and $-\frac{1}{3}$.

SOLUTION. By the methods of Sec. 10.2, we readily find the four arithmetic means between 7 and -3 to be 5, 3, 1, and -1. Hence, from the reciprocals of these numbers, we have the four required harmonic means, $\frac{1}{5}$, $\frac{1}{3}$, 1, and -1.

The single harmonic mean between two given numbers is called their *harmonic mean*. Let *H* be the harmonic mean of two given numbers *a* and *b* so that *a*, *H*, *b* are in harmonic progression. Hence, $\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ are in arithmetic progression for whose common difference we have

$$\frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H},$$

whence

$$\frac{2}{H} = \frac{1}{a} + \frac{1}{b} = \frac{b + a}{ab},$$

$$\frac{H}{2} = \frac{ab}{b + a}$$

and

$$H = \frac{2ab}{a + b}.$$

We have previously seen (Sec. 10.2) that the arithmetic mean A of two given numbers a and b is given by

$$A = \frac{a + b}{2}.$$

Hence, to avoid a very common error the student should note that the harmonic mean of two given numbers is *not* equal to the reciprocal of their arithmetic mean.

There are several interesting relations among the arithmetic, geometric, and harmonic means of two given numbers. For example, the student may readily show that the geometric mean of two numbers is also the geometric mean of their arithmetic and harmonic means. It is also left as an exercise to the student to show that, for two unequal positive numbers, their arithmetic mean is greater than their geometric mean, which, in turn, is greater than their harmonic mean. For convenience these properties of the various means are recorded in the following theorem.

Theorem 3. *Let A, G, and H represent, respectively, the arithmetic, geometric, and harmonic means of two given unequal positive numbers a and b. Then*

$$A = \frac{a + b}{2}, \qquad G = \sqrt{ab}, \qquad H = \frac{2ab}{a + b},$$

and A, G, and H are connected by the two relations

$$G^2 = AH, \qquad A > G > H.$$

EXERCISES. GROUP 34

1. The third term of a harmonic progression is $\frac{4}{3}$ and the sixth term is $\frac{2}{3}$. Find the ninth term.

2. The second term of a harmonic progression is 3 and the fifth term is $\frac{6}{5}$. Find the eighth term.

3. The first three terms of a harmonic progression are $\frac{1}{3}, \frac{3}{8}, \frac{3}{7}$. Find the sixth and eighth terms.

4. The first three terms of a harmonic progression are $-\frac{1}{3}, -1, 1$. Find the ninth term.

5. Insert three harmonic means between 2 and 1.

6. Insert four harmonic means between $-\frac{1}{2}$ and $\frac{1}{13}$.

7. Insert five harmonic means between 7 and 1.

8. Find the harmonic mean of 3 and 9.

9. Find the harmonic mean of $x + y$ and $x - y$.

10. The arithmetic mean of two numbers is 8 and their harmonic mean is 6. Find the numbers.

11. The harmonic mean of two numbers is $\frac{15}{4}$ and their arithmetic mean is 4. Find the numbers.

12. The harmonic mean of two numbers is $\frac{24}{5}$ and their geometric mean is 6. Find the numbers.

13. Determine the value of x so that the three quantities x, $x - 6$, and $x - 8$ may be in harmonic progression.

14. Determine the values of x and y if x, 4, y are in arithmetic progression and y, 3, x are in harmonic progression.

15. Three numbers are in harmonic progression, the third being equal to twice the first. If the first number is diminished by 1, the second is increased by $\frac{1}{3}$, and the third is increased by 5, the results are in geometric progression. Find the three numbers.

16. For Theorem 3 (Sec. 10.4), prove that $G^2 = AH$.

17. For Theorem 3 (Sec. 10.4), prove that $A > G > H$.

18. If a, b, c are in arithmetic progression and b, c, d are in harmonic progression, show that $ad = bc$.

19. If H is the harmonic mean of a and b, show that

$$\frac{1}{H - a} + \frac{1}{H - b} = \frac{1}{a} + \frac{1}{b}.$$

20. If a^2, b^2, c^2 are in arithmetic progression, show that $b + c$, $c + a$, $a + b$ are in harmonic progression.

21. If a, b, c are in harmonic progression, show that

$$\frac{a}{b + c}, \frac{b}{a + c}, \frac{c}{a + b}$$

are also in harmonic progression.

22. If a, b, c are in harmonic progression, show that

$$\frac{2a - b}{2}, \frac{b}{2}, \frac{2c - b}{2}$$

are in geometric progression.

23. If a, b, c are in harmonic progression, show that a, $a - c$, $a - b$ are also in harmonic progression.

24. If a is the arithmetic mean of b and c, and if b is the geometric mean of a and c, show that c is the harmonic mean of a and b.

25. If a, b, c are in harmonic progression, show that

$$\frac{a}{b + c - a}, \frac{b}{c + a - b}, \quad \text{and} \quad \frac{c}{a + b - c}$$

are also in harmonic progression.

10.5. INFINITE GEOMETRIC PROGRESSION

Heretofore we have considered only finite progressions. We will now investigate an infinite geometric progression, that is, one in which the number of terms is unlimited. For this purpose we will require an understanding of the term *limit*, a concept of fundamental importance in mathematics. Accordingly, we lay down the following

Definition. The variable x is said to approach the constant k as a *limit* provided that the numerical or absolute value of their difference, $|x - k|$, becomes and remains less than any preassigned positive quantity, however small.

The limiting process thus defined is conveniently indicated by the notations

$$\lim x = k \quad \text{or} \quad x \to k,$$

the first being read "the limit of x is k" and the second as "x approaches k as a limit."

NOTE 1. There is a more precise definition of limit than that given above: it will be encountered by the student when he later begins his study of the calculus. However, the present definition will suffice for our purposes.

The student has previously seen illustrations of the limit concept in elementary mathematics. Thus, consider the perimeter of a polygon inscribed in a given circle. If we keep increasing the number of sides, the perimeters of the resulting polygons approach closer and closer to the circumference of the circle. That is, by increasing the number of sides sufficiently, we can make the numerical value of the difference between the perimeter and the circumference less than any preassigned positive quantity, however small. In this example, the variable x of our definition is represented by the various values of the perimeter P, and the constant k is represented by the circumference C. We may then write

$$\lim P = C. \tag{1}$$

We next consider the situation where a variable increases without limit. A simple example is the case of any infinite sequence where the number of terms n is unlimited. We indicate this fact by writing $n \to \infty$, which is read "n increases without limit."

NOTE 2. It is important for the student to understand that the symbol ∞ is *not* a number. Although it is correct to read the notation $x \to 1$ as "x approaches 1," the student should avoid the tendency to read the notation $n \to \infty$ as "n approaches infinity." This notation is correctly read as stated above; it may also be read as "n increases beyond bound" or as "n becomes infinite."

The limit concept is very often associated with the variations of two related variables. For example, consider the functional relation

$$y = \frac{1}{1 + x}.$$

Suppose that x approaches 1 as a limit. It is then easy to see that y approaches $\frac{1}{2}$ as a limit. We then write

$$\lim_{x \to 1} \frac{1}{1 + x} = \frac{1}{2},$$

which is read "the limit of $1/(1 + x)$, as x approaches 1, is $\frac{1}{2}$."

In our previous example leading to relation (1) above, let P_n represent the perimeter of a polygon of n sides inscribed in a given circle of circumference C. If we now increase the number of sides n without limit, we may write

$$\lim_{n \to \infty} P_n = C,$$

which describes the limiting process more precisely than relation (1).

Let us now consider the functional relation

$$y = \frac{1}{1 - x}.$$

Suppose that x approaches 1 as a limit. It is then evident that y increases; in fact, by taking x sufficiently close to 1 we can make y exceed any assigned number, however large. We then say that as x approaches 1, y increases without limit and indicate this situation by writing

$$\lim_{x \to 1} \frac{1}{1 - x} = \infty.$$

Paradoxical as it may appear, this statement is a symbolic way of saying there is *no* limit. As previously noted, the symbol ∞ is not a number, and the statement means that as x approaches 1 in value, the expression $1/(1 - x)$ increases without limit or becomes infinite.

We will now consider several limits which are required for the infinite geometric progression whose first term is a_1 and whose common ratio is r. First, suppose that r is numerically greater than 1, that is, $|r| > 1$. Then if p is a positive number, we may write

$$|r| = 1 + p,$$

whence

$$|r^n| = (1 + p)^n = 1 + np + P,$$

where P is a positive number representing the sum of the remaining terms of the binomial expansion (Sec. 7.4). Now if n increases without limit,

np increases without limit, and hence, from the last relation above,

$$\lim_{n\to\infty} |r^n| = \infty,$$

and also

(2) $$\lim_{n\to\infty} |a_1 r^n| = \infty, \qquad |r| > 1.$$

Next, consider that $|r| < 1$, that is, $-1 < r < 1$. Then if p is a positive number, we may write

$$|r| = \frac{1}{1 + p}$$

whence $$|r^n| = \frac{1}{(1 + p)^n} = \frac{1}{1 + np + P} < \frac{1}{np}.$$

Now if n increases without limit, np increases without limit, and $1/np$ approaches the limit zero. Hence, from the last relation

$$\lim_{n\to\infty} |r^n| = 0,$$

and also

(3) $$\lim_{n\to\infty} |a_1 r^n| = 0, \qquad |r| < 1.$$

From Theorem 2 (Sec. 10.3), for a geometric progression whose first term is a_1 and whose common ratio is r, the sum of the first n terms is given by

(4) $$s_n = \frac{a_1(1 - r^n)}{1 - r}, \qquad r \neq 1.$$

We are now interested in the effect on s_n produced by increasing the number of terms n without limit, thus giving us an *infinite geometric series* (Sec. 10.1, Note). We shall see that under certain conditions, s_n may increase without limit or may oscillate between two values; the series is then said to *diverge*. However, under other conditions s_n may approach a definite limit; this limit is *defined* as the *sum* of the series which is then said to *converge*.

Thus, consider the infinite geometric series

(5) $$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} + \cdots.$$

We may easily show by means of the formula (4) that for n terms of this series,

$$s_n = 2 - \frac{1}{2^{n-1}}.$$

Now as n increases, $1/(2^{n-1})$ approaches zero and s_n approaches 2. We represent this by the statement

$$s_\infty = \lim_{n\to\infty} s_n = 2,$$

that is, the limiting sum of the infinite series (5) is 2, and hence the series is *convergent.* Note that the word "sum" here does not have the ordinary meaning of a sum as the result of adding a finite number of quantities.

NOTE 3. The student will find it an interesting exercise to illustrate series (5) geometrically by considering the line segment of length 2 units and having its end points at the origin O and the point P on the positive X-axis. Let P_1 be the midpoint of OP, P_2 the midpoint of P_1P, P_3 the midpoint of P_2P, and so on. Then interpret the meaning of the length of OP_n and the position of the point P_n as n takes on the values $1, 2, 3, \cdots$.

We will now consider the effect on s_n for various values of the ratio r. Formula (4) does not define s_n for $r = 1$. In this case the series assumes the form:

$$s_n = a_1 + a_1 + a_1 + \cdots = na_1,$$

whence

$$s_\infty = \lim_{n\to\infty} s_n = \infty,$$

that is, s_n increases without limit and the series is *divergent.*

If $r = -1$, the series assumes the form

$$s_n = a_1 - a_1 + a_1 - a_1 + \cdots.$$

If n is odd, $s_n = a_1$; if n is even, $s_n = 0$. Since this is true for all values of n, it follows that s_n oscillates between a_1 and 0. Such a series is termed an *oscillating series*; it is divergent.

We now consider values of the ratio r other than ± 1. For this purpose we write formula (4) in the form:

(6) $$s_n = \frac{a_1}{1-r} - \frac{a_1r^n}{1-r}, \qquad r \neq 1.$$

If $|r| > 1$, it follows from (2) that $\lim_{n\to\infty} \frac{a_1r^n}{1-r} = \infty$, whence, from (6),

$$s_\infty = \lim_{n\to\infty} s_n = \infty,$$

and the series is divergent.

Finally, we consider the case when $|r| < 1$. From (3) it follows that $\lim_{n\to\infty} \frac{a_1r^n}{1-r} = 0$ whence, from (6),

$$s_\infty = \lim_{n\to\infty} s_n = \frac{a_1}{1-r}, \qquad |r| < 1,$$

and the series is convergent.

We record the preceding results as

Theorem 4. *The infinite geometric series whose first term is a_1 and whose common ratio is r is convergent for all values of r such that $|r| < 1$, and its sum is given by the relation*

$$s_\infty = \lim_{n\to\infty} s_n = \frac{a_1}{1-r}, \qquad |r| < 1,$$

where n represents the number of terms.

For all other values of r, the series is divergent.

Example 1. A rubber ball is dropped from a height of 16 ft. If it rebounds one fourth of the height from which it falls, find the limiting value of the total distance traversed by the ball before theoretically coming to rest.

SOLUTION. Actually, due to resistance, the ball will come to rest in a finite time, but after a sufficient number of bounces, the distance traversed will be very close to the limiting value.

The limiting value of the total distance traversed is the sum of two infinite geometric progressions:

Falls: $16, 4, 1, \frac{1}{4}, \cdots,$

Rebounds: $4, 1, \frac{1}{4}, \cdots.$

By Theorem 4, this limiting value is given by

$$\frac{16}{1-\frac{1}{4}} + \frac{4}{1-\frac{1}{4}} = \frac{64}{3} + \frac{16}{3} = 26\tfrac{2}{3} \text{ ft.}$$

An example of an infinite geometric progression is a repeating decimal. Such a decimal is nonterminating and, from some point on, one digit or a group of digits is repeated without end. Thus, examples of repeating decimals and their equivalent infinite geometric series are

$$0.333\cdots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots,$$

and

$$2.151515\cdots = 2 + \frac{15}{10^2} + \frac{15}{10^4} + \frac{15}{10^6} + \cdots.$$

A repeating decimal may be abbreviated by placing dots over the repeated digits. Thus, the two examples above may be written as $0.\dot{3}$ and $2.\dot{1}\dot{5}$, respectively. It may be proved that every repeating decimal represents a rational number.

Example 2. Find the rational fraction equivalent to the repeating decimal $1.2\dot{6}$.

SOLUTION. We set aside 1.2, which is not repeated. Then for $0.0\dot{6}$ we have

$$0.0\dot{6} = \frac{6}{10^2} + \frac{6}{10^3} + \frac{6}{10^4} + \cdots,$$

where $a_1 = 0.06$ and $r = 0.1$.

Hence,
$$s_\infty = \frac{a_1}{1 - r} = \frac{0.06}{1 - 0.1} = \frac{6}{90} = \frac{1}{15},$$

and
$$1.2\dot{6} = 1.2 + \frac{1}{15} = \frac{6}{5} + \frac{1}{15} = \frac{18}{15} + \frac{1}{15} = \frac{19}{15}.$$

This result is readily checked by actual division.

EXERCISES. GROUP 35

In each of Exs. 1–6, find the (limiting) sum of the given infinite geometric progression.

1. $12, 6, 3, \cdots$.
2. $9, -3, 1, \cdots$.
3. $3, \sqrt{3}, 1, \cdots$.
4. $\sqrt{2}, 2 - \sqrt{2}, 3\sqrt{2} - 4, \cdots$.
5. $\sqrt{5}, 1, \frac{\sqrt{5}}{5}, \cdots$.
6. $1, \frac{1}{1 + x}, \frac{1}{(1 + x)^2}, \cdots, x > 0$.

In each of Exs. 7–14, find the rational fraction equivalent to the given repeating decimal, and check the result.

7. $0.\dot{7}$.
8. $2.\dot{5}$.
9. $0.\dot{3}\dot{5}$.
10. $1.\dot{2}\dot{1}$.
11. $0.\dot{1}\dot{2}\dot{3}$.
12. $3.\dot{2}\dot{0}\dot{1}$.
13. $0.45\dot{1}\dot{2}$.
14. $1.\dot{0}\dot{3}\dot{7}$.

15. Verify the following method for obtaining the rational number equivalent to a given repeating decimal in which only one digit is repeated. Let x = the given repeating decimal. Multiply by 10, thus obtaining $10x$ = 10 times the given decimal. Subtract the first relation from the second, thus obtaining $9x$ = a terminating decimal. Divide by 9 and simplify if necessary; the result is the required rational number. Illustrate the method by solving Ex. 7.

16. Extend the method of Ex. 15 to the case of a repeating decimal in which two digits are repeated. Illustrate the method by solving Ex. 9.

17. Extend the method of Ex. 15 to the case of a repeating decimal in which three digits are repeated. Illustrate the method by solving Ex. 14.

18. A rubber ball dropped from a height of 27 ft rebounds one third of the height from which it falls. Find the limiting value of the total distance traversed by the ball before theoretically coming to rest.

19. A rubber ball dropped from a height of 25 ft rebounds one fifth of the height from which it falls. Find the limiting value of the total distance traversed by the ball before theoretically coming to rest.

20. The bob of a pendulum moves through a distance of 8 in. during the first swing. In each following swing, the bob moves through $\frac{7}{8}$ of the distance of the preceding swing. Find the limiting value of the total distance traversed by the bob before theoretically coming to rest.

21. The limiting sum of an infinite geometric series is $21\frac{1}{3}$. If the first term is 16, find the fifth term.

22. The limiting sum of an infinite geometric series is 81. If the ratio is $\frac{2}{3}$, find the seventh term.

23. From the definition of limit, show that any constant may be regarded as its own limit.

24. By actual division, show that $a_1/(1 - r)$ gives an infinite geometric series whose first term is a_1 and whose common ratio is r.

25. In a convergent infinite geometric series, show that the nth term approaches zero as $n \to \infty$. *Note.* This is a necessary but not a sufficient condition for the convergence of any infinite series.

26. Carry out the details of the exercise described in Note 3 (Sec. 10.5).

27. For the infinite geometric series $1 + \frac{1}{2} + \frac{1}{4} + \cdots$, determine the least number of terms whose sum differs from 2 by less than 0.001.

28. If a, b, c are in arithmetic progression, show that $a^2(b + c)$, $b^2(c + a)$, $c^2(a + b)$ are in arithmetic progression.

29. If $(a - b)/(b - c) = a/x$, show that a, b, c are in arithmetic, geometric, or harmonic progression, according as $x = a$, b, or c, respectively.

30. If a, b, c are in arithmetic progression, b, c, d are in geometric progression, and c, d, e are in harmonic progression, show that a, c, e are in geometric progression.

11

Theory of equations

11.1. INTRODUCTION

We have now arrived at a very important stage in algebra, for we are about to consider the problem of determining the roots of algebraic equations of any degree—one of the primary objectives in the study of algebra. In particular, this chapter will confine itself to the rational integral equation of degree n:

$$(1) \qquad a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0, \qquad a_0 \neq 0,$$

where n is a positive integer and the coefficients $a_0, a_1, \cdots, a_n$ are any constants. It is convenient to refer to a_0, the coefficient of the term of highest degree, as the *leading coefficient.*

For $n = 1$, equation (1) is the *linear equation* studied in Chapter 4; for $n = 2$, equation (1) is the *quadratic equation* studied in Chapter 5. Hence, in this chapter, we shall consider equations of type (1) for which $n \geq 3$.

We have seen previously that the solutions of linear and quadratic equations may be expressed in terms of their coefficients by means of a finite number of one or more of the six operations of algebra (Sec. 4.4, Theorem 1; Sec. 5.4, Theorem 1). Such a solution is called an *algebraic solution* (Sec. 1.6, Fundamental Definition); it is also called a *solution by radicals.* There are algebraic solutions of equation (1) for $n = 3$, the cubic equation, and for $n = 4$, the quartic or biquadratic equation; these solutions, however, are somewhat involved and not very practical for ordinary use. Consequently, they will not be considered here. Furthermore, it is shown in advanced treatises that for $n \geq 5$, the general rational integral equation (1) has no algebraic solution. (See Sec. 3.6, Note 2.)

Since we do not intend to use the algebraic solutions of equation (1) for $n = 3$ and $n = 4$, and since there is no algebraic solution for $n \geq 5$, the question naturally arises, how do we propose to solve an equation of degree higher than 2? Frankly, the answer is that we attempt to guess or approximate to a desired degree of accuracy the values of the roots and then prove or disprove our conjectures by actual substitution in the given equation. Because the root of an equation may range over an unlimited number of values, it is evident that if this method is to be of any practical use, we must be able to confine the field of our conjectures within reasonable limits. Thus, in the sections which follow, we shall show how it is possible, under certain conditions, to determine the number, nature, and possible values of the roots before attempting the complete solution of a given equation.

11.2. THE GENERAL PROBLEM

It appears from the preceding section that the full discussion of the properties and solution of the general rational integral equation is a problem of considerable magnitude. In fact, entire treatises are devoted exclusively to the theory of equations. Having only a single chapter at our disposal, we can merely give an introduction to this fascinating subject. However, we shall select those topics which will be most useful to the student, both for his present mathematical needs and those of the immediate future. Later, after further training, particularly in the calculus, the student will be in a position to pursue this subject further in advanced treatises.

In this section we shall indicate briefly the nature and scope of this chapter. While the coefficients in the general equation are constants and as such may be any numbers, real or complex, we shall, in general, consider the solution of only those equations whose coefficients are real. Furthermore, for any given equation we shall confine our attention initially to the determination of any real roots, rational and irrational, and then, if possible, find any complex roots by previous methods. In the sections which follow, each theorem and procedure considered is presented with these objectives in view.

For convenience, it shall be understood hereafter that the general rational integral equation (1) of Art. 11.1 will be represented by the equation $f(x) = 0$, where the left member $f(x)$ is the polynomial in x of degree n.

11.3. THE REMAINDER AND FACTOR THEOREMS

We now consider a simple but extremely important result known as the *remainder theorem*. Before formally stating and proving this theorem, we will illustrate its meaning by an example. Thus, if we divide the polynomial $f(x) = 3x^3 - 4x^2 - 2x - 7$ by $x - 2$, using ordinary algebraic division (Sec. 2.7), we obtain a quotient $Q(x) = 3x^2 + 2x + 2$ and a remainder $R = -3$. Also, if we substitute 2 for x in the original dividend $f(x)$, we find $f(2) = 3(2)^3 - 4(2)^2 - 2(2) - 7 = -3$. The fact that $f(2)$ and the remainder R are both equal to -3 may, of course, be merely a coincidence in this particular case. But we shall now show that this result is true in every case by establishing

Theorem 1. (*Remainder Theorem*). *If the polynomial $f(x)$ is divided by $x - r$, where r is a constant free of x, the remainder is equal to $f(r)$.*

PROOF. We first write the polynomial $f(x)$ in the form

$$(1) \qquad f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n, \qquad a_0 \neq 0.$$

Then

$$(2) \qquad f(r) = a_0r^n + a_1r^{n-1} + \cdots + a_{n-1}r + a_n.$$

Subtracting (2) from (1), member by member, we have

$$(3) \quad f(x) - f(r) = a_0(x^n - r^n) + a_1(x^{n-1} - r^{n-1}) + \cdots + a_{n-1}(x - r).$$

Now it may be shown by mathematical induction that for every positive integral value of n, $x^n - r^n$ is exactly divisible by $x - r$. (See Ex. 6, Group 24, Sec. 7.3.) Hence, from (3), it follows that $f(x) - f(r)$ is exactly divisible by $x - r$. Say that as a result of such division, the quotient obtained is the polynomial $Q(x)$. Then we may write

$$f(x) - f(r) = (x - r)Q(x),$$

whence

$$f(x) = (x - r)Q(x) + f(r),$$

and

$$\frac{f(x)}{x - r} = Q(x) + \frac{f(r)}{x - r},$$

that is, the division of $f(x)$ by $x - r$ gives a remainder equal to $f(r)$, as was to be shown.

By means of the remainder theorem we can establish another important and useful result given by

Theorem 2 (*Factor Theorem*). *If r is a root of the rational integral equation $f(x) = 0$, then $x - r$ is a factor of the polynomial $f(x)$, and conversely.*

PROOF. Since r is a root of $f(x) = 0$, it follows by the definition of a root that $f(r) = 0$. But by the remainder theorem, the division of $f(x)$ by

$x - r$ gives a remainder $R = f(r)$. Hence $R = 0$, that is, the division is exact, and $x - r$ is a factor of $f(x)$.

Conversely, suppose that $x - r$ is a factor of $f(x)$. Then $x - r$ is an exact divisor of $f(x)$ and the remainder $R = 0$. Hence, by the remainder theorem, $R = f(r) = 0$, and r is a root of $f(x) = 0$.

NOTE. We now see that Theorem 4 (Sec. 5.5) for a quadratic equation is a special case of Theorem 2.

Example 1. Without actually performing the division, find the remainder when the polynomial $f(x) = x^4 + 5x^3 + 5x^2 - 4x - 7$ is divided by $x + 3$.

SOLUTION. By the remainder theorem, the remainder obtained by dividing the given polynomial $f(x)$ by $x + 3$ is $f(-3) = (-3)^4 + 5(-3)^3 + 5(-3)^2 - 4(-3) - 7 = 81 - 135 + 45 + 12 - 7 = -4$.

The student may readily verify this result by actually performing the division.

Example 2. By means of the factor theorem, show that $x - 5$ is a factor of $f(x) = x^3 - 8x^2 + 19x - 20$.

SOLUTION. For $x - 5$ to be a factor of $f(x)$, we must have $f(5) = 0$. Thus, $f(5) = 5^3 - 8 \cdot 5^2 + 19 \cdot 5 - 20 = 125 - 200 + 95 - 20 = 0$.

Example 3. By means of the remainder theorem, prove that $x^n - a^n$ is exactly divisible by $x - a$ for every positive integral value of n.

SOLUTION. By the remainder theorem, if $f(x) = x^n - a^n$ is divided by $x - a$, the remainder is $f(a) = a^n - a^n = 0$, and the division is exact.

This result may also be obtained by mathematical induction. (See Exs. 6 and 7 of Group 24, Sec. 7.3.)

11.4. SYNTHETIC DIVISION

As we have seen in the preceding section, the remainder theorem enables us to evaluate the polynomial $f(x)$ for specified values of x without actually substituting. Since this requires the division of a polynomial by a binomial, the process might be somewhat lengthy by ordinary division. This operation, however, may be performed rapidly by an abbreviated form of division known as *synthetic division*. We will illustrate the procedure by considering the division of the polynomial $3x^3 - 4x^2 - 2x - 7$ by $x - 2$.

By ordinary algebraic division (Sec. 2.7), this operation appears as follows:

$$
\begin{array}{r|l}
 & 3x^2 + 2x + 2 \text{ (Quotient)} \\
\hline
x - 2 & 3x^3 - 4x^2 - 2x - 7 \\
 & \underline{3x^3 - 6x^2} \\
 & \qquad\quad 2x^2 - 2x \\
 & \qquad\quad \underline{2x^2 - 4x} \\
 & \qquad\qquad\quad 2x - 7 \\
 & \qquad\qquad\quad \underline{2x - 4} \\
 & \qquad\qquad\qquad - 3 \text{ (Remainder)}
\end{array}
$$

It is now our object to abridge, as much as possible, the work above. Since all polynomials are arranged according to descending powers of x, we may omit such powers and retain only their coefficients. Furthermore, since the coefficient of x in the divisor is unity, the first term of each partial product is a repetition of the term immediately above it, and hence may be omitted. Also, since the second term of each partial remainder is a repetition of the term above it in the dividend, it may be omitted. For convenience, we omit the first term of the divisor and place its constant term at the right of the dividend. In addition, since each coefficient of the quotient, with the exception of the first, is represented by the first coefficient of the partial remainder, the entire quotient may be omitted. As a result of these omissions, the division above now appears as follows:

$$
\begin{array}{cccc|l}
3 & -4 & -2 & -7 & \underline{-2} \\
 & \underline{-6} & & & \\
 & 2 & & & \\
 & & -4 & & \\
\cline{1-3}
 & & 2 & & \\
 & & & \underline{-4} & \\
 & & & -3 &
\end{array}
$$

For compactness, we now arrange the work in three lines as follows, repeating the leading coefficient in the third line:

$$
\begin{array}{cccc|l}
3 & -4 & -2 & -7 & \underline{-2} \\
 & -6 & -4 & -4 & \\
\cline{1-4}
3 & +2 & +2 & -3 &
\end{array}
$$

If we change the sign of the term in the divisor, we may *add* the partial products instead of subtracting them. This is desirable, for the remainder obtained as the result of this division is the value of $f(x)$ when 2, not -2, is substituted for x. Accordingly, the final form of our division is exhibited as follows:

$$\begin{array}{r|l} 3 - 4 - 2 - 7 & \underline{2} \\ \underline{+ 6 + 4 + 4} & \\ 3 + 2 + 2 \,|\, \underline{-3} & \end{array}$$

From the third line the quotient is $3x^2 + 2x + 2$ and the remainder, separated from the quotient as shown, is -3.

For convenient reference, we give here the

Rule for Synthetic Division

To divide a polynomial $f(x)$ by $x - r$, proceed as follows:

In the first line, write in order the coefficients $a_0, a_1, a_2, \cdots, a_n$ of the dividend $f(x)$, and write the number r separately at the right. If any power of x is missing in $f(x)$, write its coefficient as zero.

Write the leading coefficient a_0 in the first place in the third line, multiply it by r, and write the product a_0r in the second line under a_1. Add a_1 and the product a_0r and write the sum $a_1 + a_0r$ in the third line. Multiply this sum by r, write the product in the second line under a_2 and add it to a_2, and write the sum in the third line. Continue this process until finally a product has been added to a_n and the sum written in the third line.

The last number in the third line is the remainder; the preceding numbers in the third line are the coefficients of the quotient when arranged in the order of descending powers of x.

NOTE. That the rule given above holds in general may be established by mathematical induction.

After a little practice, the student will be able to perform the operation of synthetic division with great rapidity. We illustrate the rule in the following

Example. By synthetic division, find the quotient and remainder when $2x^4 + 3x^3 - x - 3$ is divided by $x + 2$.

SOLUTION. We first note that because the dividend lacks the term in x^2, we must supply a zero coefficient for its place. Furthermore, since we are dividing by $x + 2 = x - (-2) = x - r$, we must take $r = -2$. The work then appears as follows:

$$\begin{array}{r|l} 2 + 3 + 0 - 1 - \ 3 & \underline{-2} \\ \underline{- 4 + 2 - 4 + 10} & \\ 2 - 1 + 2 - 5 \,|\, \underline{+7} & \end{array}$$

Hence, the quotient is $2x^3 - x^2 + 2x - 5$ and the remainder is 7.

EXERCISES. GROUP 36

In each of Exs. 1–4, by means of the remainder theorem, prove the given statement, n being a positive integer.

1. $x^n - a^n$ is exactly divisible by $x + a$ if n is even.
2. $x^n + a^n$ is exactly divisible by $x + a$ if n is odd.
3. $x^n + a^n$ is not exactly divisible by $x + a$ if n is even.
4. $x^n + a^n$ is not exactly divisible by $x - a$ if n is even.

In each of Exs. 5–10, for the given polynomial, find the indicated values, using synthetic division and the remainder theorem.

5. $f(x) = 2x^3 - 3x^2 + 5x - 7;\ f(2), f(-1)$.
6. $f(x) = 3x^4 - 5x^3 + 2x^2 - 7x + 8;\ f(1), f(-2)$.
7. $f(x) = x^5 - 2x^4 - 3x^2 - 2x - 8;\ f(3), f(-1)$.
8. $f(x) = 2x^3 - 3x^2 + 3x - 2;\ f(\frac{1}{2}), f(-\frac{1}{2})$.
9. $f(x) = 9x^4 - 3x^2 + 2x - 1;\ f(\frac{1}{3}), f(0.1)$.
10. $f(x) = x^3 - 2x^2 + 3x - 2;\ f(0.2), f(-0.1)$.

In each of Exs. 11–15, for the indicated operation, find the quotient and the remainder, using synthetic division.

11. $(x^3 + 4x^2 + 7x - 2) \div (x + 2)$.
12. $(x^4 + 2x^3 - 10x^2 - 11x - 7) \div (x - 3)$.
13. $(x^6 - x^4 + x^2 - 2) \div (x - 1)$.
14. $(2x^5 - 14x^3 + 8x^2 + 7) \div (x + 3)$.
15. $(4x^4 - 3x^2 + 3x + 7) \div (x + \frac{1}{2})$.

In each of Exs. 16–20, using the factor theorem and synthetic division, determine whether the given binomial is a factor of the given polynomial.

16. $x - 1;\ f(x) = x^3 + 2x^2 - 4x + 1$.
17. $x + 2;\ f(x) = x^4 - 3x^3 - 2x^2 + 5x - 9$.
18. $x + 3;\ f(x) = x^5 + 4x^4 - 7x^2 + 5x - 3$.
19. $x - 5;\ f(x) = x^4 - 5x^3 - x + 5$.
20. $x - 2;\ f(x) = x^6 - 5x^5 + 3x^3 - x^2 + 7$.

In each of Exs. 21–25, using the factor theorem and synthetic division, determine whether the given equation has the root shown.

21. $x^3 - 9x^2 + 26x - 24 = 0;\ x = 2$.
22. $x^4 + 5x^3 + 4x^2 - 7x - 3 = 0;\ x = -3$.
23. $2x^4 + 10x^3 + 11x^2 - 2x + 5 = 0;\ x = -2$.
24. $3x^5 - x^4 + 2x^3 - 4x^2 + 3x - 10 = 0;\ x = 1$.
25. $5x^6 + 3x^5 - 2x^3 - 7x^2 + 1 = 0;\ x = 1$.

In each of Exs. 26–30, use the factor theorem and synthetic division to obtain the required result.

26. Show that $x - 3$ is a factor of $x^3 - 2x^2 - 23x + 60$, and find the remaining factors.

27. Show that $x - 1$ and $x + 2$ are factors of $x^4 + 2x^3 - 7x^2 - 8x + 12$, and find the remaining factors.

28. By trial, find all the real factors of $x^4 - x^3 - 4x^2 - 5x - 3$.

29. Show that two of the roots of $x^4 + x^3 - 16x^2 - 4x + 48 = 0$ are 2 and -4, and find the remaining roots.

30. By trial, find all the roots of $x^4 - x^3 - 2x^2 - 2x + 4 = 0$.

31. Using synthetic division, find the quotient and remainder when $2x^4 - 5x^3 + 3x^2 - x + 3$ is divided by $2x + 1$. *Hint:* Divide synthetically by $x + \frac{1}{2}$ and then divide the quotient by 2.

32. Using synthetic division, find the quotient and remainder when $3x^4 + 2x^3 + 5x^2 - 5x - 3$ is divided by $3x - 1$.

33. By the remainder theorem, find the value of k so that the polynomial $3x^3 - 2x^2 + kx - 8$ is exactly divisible by $x - 2$.

34. Find the value of k so that the polynomial $2x^3 + kx^2 - 3x - 4$ is exactly divisible by $x + 1$.

35. Find the value of k so that when $x^4 + 2x^3 - 3x^2 + kx - 7$ is divided by $x - 2$, the remainder is equal to 3.

36. Find the value of k so that when $4x^3 + kx^2 - 2x + 5$ is divided by $x - 1$, the remainder is equal to 5.

37. Find the values of a and b if $x - 1$ and $x + 2$ are factors of the polynomial $x^4 + ax^3 + bx - 2$.

38. Find the values of a and b if 2 and -3 are roots of the equation $x^4 + x^3 + ax^2 + bx + 30 = 0$.

39. Show that the rational integral equation $f(x) = 0$ has the root $x = 1$ if the sum of its coefficients is equal to zero.

40. Show by mathematical induction that the rule for synthetic division (Sec. 11.4) holds in general.

11.5. THE POLYNOMIAL GRAPH

We have previously considered the graphical representation of algebraic functions (Secs. 3.9, 9.4) and have seen its many advantages. For this reason we now study the general problem of the construction and interpretation of the graph of the polynomial $f(x)$. We recall (Sec. 3.9) that, as shown in the calculus, this graph is a smooth continuous curve. This fact, as we shall soon see, is of great value in locating the real zeros of $f(x)$ and hence the real roots of the equation $f(x) = 0$ (Sec. 4.2). Accordingly, we now consider the following example.

Example 1. Construct the graph of the polynomial

(1) $$f(x) = x^4 - x^3 - 12x^2 + 8x + 24,$$

and use it to locate any real roots of the equation $f(x) = 0$.

SOLUTION. We first obtain the coordinates of a suitable number of points for the graph. Heretofore this has been done by substituting assigned values of x directly in $f(x)$. However, in many cases such coordinates may be obtained with far less labor by the use of synthetic division. We shall see here, and later, additional advantages in using synthetic division.

The first question that arises is what values to assign to x. Initially, it is generally convenient to let x take on the values 0, ± 1, ± 2, and so on, continuing this process only so long as it gives us useful information about any real roots. Thus, for the function (1), we have the following pairs of corresponding values:

x	0	1	2	3	4	-1	-2	-3	-4
$f(x)$	24	20	0	-6	56	6	-16	0	120

The reasons for not continuing beyond $x = \pm 4$ will be apparent from the synthetic division for $x = 4$ and $x = -4$, as shown below.

$$\begin{array}{rrrrr|l} 1 & -1 & -12 & +8 & +24 & \underline{4} \\ & +4 & +12 & +0 & +32 & \\ \hline 1 & +3 & +0 & +8 & +56 & \end{array} \qquad \begin{array}{rrrrr|l} 1 & -1 & -12 & +8 & +24 & \underline{-4} \\ & -4 & +20 & -32 & +96 & \\ \hline 1 & -5 & +8 & -24 & +120 & \end{array}$$

For $x = 4$, all the numbers in the third line of the synthetic division are positive or zero. Hence for a value of $x > 4$, the remainder will be positive and greater than 56; therefore there is no real zero greater than 4.

Similarly, for $x = -4$, all the numbers in the third line of the synthetic division are alternately positive and negative. Hence for a value of $x < -4$, the remainder will be positive and greater than 120; therefore there is no real zero less than -4.

It is evident from our table of values that 2 and -3 are zeros of $f(x)$ and thus roots of $f(x) = 0$.

We note, furthermore, that $f(x)$ changes from a negative value (-6) to a positive value (56) as x changes from 3 to 4. Since $f(x)$ has a continuous graph, this means that $f(x)$ must assume a zero value, and therefore cross the X-axis, at least once between $x = 3$ and $x = 4$. That is, the equation $f(x) = 0$ has at least one real root between 3 and 4. By a similar argument, we see that $f(x) = 0$ has at least one real root between -1 and -2.

Plotting the points whose coordinates are given in the table of values above, and drawing a smooth curve through them, we obtain the graph shown in Fig. 37. In the next section we will see that an equation of the fourth degree has exactly four roots. Hence, in this example, we have accounted for all of the roots of $f(x) = 0$.

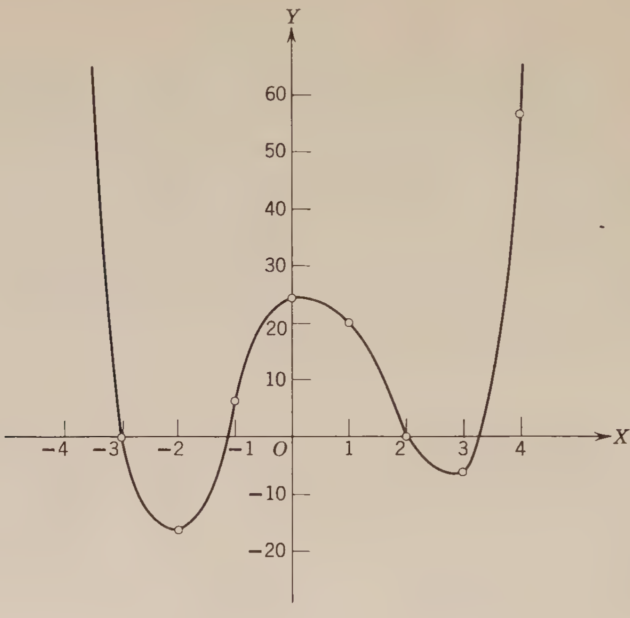


Figure 37

In the preceding example we studied an equation, all of whose roots are real and different. We will next consider an equation whose roots are not all real and are not all different.

If two of the roots of an equation are equal, we are said to have a *double root*, if three roots are equal a *triple root*, and so on. In general, such roots are said to be *repeated* or *multiple* roots. If an equation has m roots each equal to r, then r is said to be a *root of multiplicity m*. In the next example we will consider an equation having multiple and complex roots.

Example 2. Construct the graph of the polynomial

$$f(x) = (x + 1)^2(x - 2)^3(x^2 + x + 1), \tag{2}$$

and discuss the roots of $f(x) = 0$.

SOLUTION. Usually we are not given a polynomial in factored form as in (2), but it is often possible to obtain such a form by trial, using the factor theorem and synthetic division. By the methods of Chapter 5 we may show that the quadratic factor $x^2 + x + 1$ is irreducible in the field of real numbers and has the conjugate complex numbers $\frac{-1 \pm \sqrt{3}i}{2}$ as its zeros. Hence the equation $f(x) = 0$ has -1 as a double root, 2 as a triple root, and $\frac{-1 \pm \sqrt{3}i}{2}$ as conjugate complex roots.

We construct the graph of $f(x)$ in order to show the effect of multiple roots. For greater accuracy we take values of x at intervals of 0.5, as shown in the table accompanying the graph in Fig. 38.

x	$f(x)$
0	-8
0.5	-13.1
1	-12
1.5	-3.7
2	0
2.5	14.9
3	20.8
-0.5	-0.98
-1	0
-1.5	-18.8
-2	-64

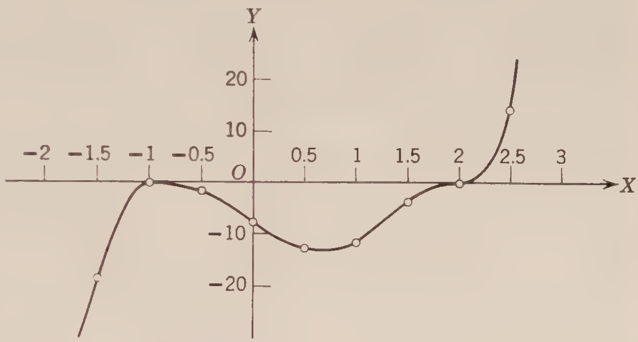


Figure 38

We note that for the double root -1, the graph is tangent to the X-axis at $x = -1$ but does not cross it; this is a characteristic of a multiple root appearing an *even* number of times. We may also show this fact by the method of inequalities (Chapter 6) applied to the polynomial $f(x)$ in the vicinity of the critical value $x = -1$. For x slightly greater or less than -1, the factor $(x + 1)^2$ is positive, the factor $(x - 2)^3$ is negative, and the quadratic factor $x^2 + x + 1$ is positive. Hence, $f(x)$ is negative and does not cross the X-axis at $x = -1$.

We note also that for the triple root 2, the graph is tangent to the X-axis at $x = 2$ and also crosses it there; this is a characteristic of a multiple root appearing an *odd* number of times. This may be shown by applying the method of inequalities to the polynomial $f(x)$ in the vicinity of the critical value $x = 2$. For x slightly less than 2, $(x + 1)^2$ is positive, $(x - 2)^3$ is negative, and $x^2 + x + 1$ is positive; hence $f(x)$ is negative. For x slightly greater than 2, $(x + 1)^2$ is positive, $(x - 2)^3$ is positive, and $x^2 + x + 1$ is positive; hence $f(x)$ is positive. Since $f(x)$ changes sign in going from the left to the right of $x = 2$, it follows from the continuity of the function that the graph must cross the X-axis at $x = 2$.

For the quadratic factor $x^2 + x + 1$, which is always positive and has no real zeros, there are no points on the X-axis.

From the discussion in these two examples, we may arrive at some important conclusions concerning the polynomial graph and its corresponding rational integral equation. For convenient reference we give a summary of these facts below. The rigorous proof of several of these statements will be found in advanced treatises.

Characteristics of the Polynomial $f(x)$ with Real Coefficients and the Rational Integral Equation $f(x) = 0$

In the synthetic division of $f(x)$ by $x - r$, where r is positive, if all the numbers in the third line are positive or zero, then $f(x) = 0$ does not have a real root greater than r.

In the synthetic division of $f(x)$ by $x - r$, where r is negative, if the numbers in the third line alternate in sign, then $f(x) = 0$ does not have a real root less than r.

If a and b are real numbers such that $f(a)$ and $f(b)$ are opposite in sign, then the graph of $f(x)$ crosses the X-axis at least once between $x = a$ and $x = b$, and the equation $f(x) = 0$ has at least one real root between a and b.

If r is a non-repeated real root of $f(x) = 0$, then the graph of $f(x)$ crosses the X-axis at $x = r$ but is not tangent at that point.

Let r be a real repeated root of $f(x) = 0$ and of multiplicity m. If m is even, the graph of $f(x)$ is tangent to the X-axis at $x = r$ but does not cross the X-axis at that point. If m is odd, the graph of $f(x)$ is tangent to the X-axis at $x = r$ and crosses the X-axis at that point.

EXERCISES. GROUP 37

In each of Exs. 1–14, construct the graph of the given polynomial and locate any real roots of the equation $f(x) = 0$.

1. $f(x) = x^3 - 6x^2 + 11x - 6$.
2. $f(x) = x^3 + 2x^2 - 5x - 6$.
3. $f(x) = x^3 - 2x^2 - 8x$.
4. $f(x) = x^3 - x^2 - x - 2$.
5. $f(x) = x^4 - 5x^2 + 4$.
6. $f(x) = x^4 - 13x^2 - 12x$.
7. $f(x) = x^4 - 3x^3 - 11x^2 + 25x + 12$.
8. $f(x) = x^4 - 2x^3 - 12x^2 + 2x + 11$.
9. $f(x) = x^4 - 3x^3 - 17x^2 + 21x + 34$.
10. $f(x) = x^3 - 4x^2 + 7x - 4$.
11. $f(x) = x^4 + 4x^3 + x^2 - 16x - 20$.
12. $f(x) = x^4 - 2x^3 - 4x^2 - 16x$.
13. $f(x) = x^5 + x^4 - 5x^3 - x^2 + 8x - 4$.
14. $f(x) = x^5 - x^4 - 8x^3 + 8x^2 + 16x - 16$.

In each of Exs. 15–18, sketch the graph of $f(x)$ without first expanding into a polynomial.

15. $f(x) = (x - 1)^2(x + 2)^3$.
16. $f(x) = x(x + 3)^3(x - 4)^2$.

17. $f(x) = (x + 2)^2(x - 1)^3(x^2 + 1)$.

18. $f(x) = (x + 2)(x - 2)^2(x - 4)^4$.

In each of Exs. 19–23, solve the given inequality.

19. $x^3 - 6x^2 + 11x - 6 > 0$.

20. $x^4 - 10x^2 + 9 > 0$.

21. $x^4 + 2x^3 - x - 2 > 0$.

22. $x^4 + x^3 - x^2 - 7x - 6 < 0$.

23. $x^5 - 2x^4 - 4x^3 + 4x^2 - 5x + 6 > 0$.

24. On the basis of the continuity of the polynomial function $f(x)$, show that, if a and b are real numbers such that $f(a)$ and $f(b)$ have the same sign, then the equation $f(x) = 0$ has either no real roots or else an even number of real roots between a and b.

25. On the basis of the continuity of the polynomial function $f(x)$, show that, if a and b are real numbers such that $f(a)$ and $f(b)$ are opposite in sign, then the equation $f(x) = 0$ has an odd number of real roots between a and b.

11.6. NUMBER OF ROOTS

We have already seen that the linear equation has exactly one root and the quadratic equation exactly two. These are special cases of the general theorem that the rational integral equation of nth degree has exactly n roots. In order to prove this theorem we need

Theorem 3. (*Fundamental Theorem of Algebra*). *A rational integral equation $f(x) = 0$ has at least one root, real or complex.*

The proof of this theorem, known as the *fundamental theorem of algebra*, requires advanced methods beyond the scope of this book. Hence we *assume* its validity in establishing

Theorem 4. *A rational integral equation $f(x) = 0$, of degree n, has exactly n roots.*

PROOF. Let the rational integral equation be represented by

(1) $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0, \ a_0 \neq 0.$

By Theorem 3, equation (1) has at least one root, say r_1. Hence by the factor theorem (Sec. 11.3, Theorem 2), $x - r_1$ is a factor of $f(x)$, and we may write

$$f(x) \equiv (x - r_1)Q_1(x),$$

where $Q_1(x)$ is a polynomial of degree $n - 1$ with a_0 as its leading coefficient.

By Theorem 3, $Q_1(x) = 0$ has at least one root, say r_2. Hence by the factor theorem, $x - r_2$ is a factor of $Q_1(x)$, and we may write

$$f(x) \equiv (x - r_1)(x - r_2)Q_2(x),$$

where $Q_2(x)$ is a polynomial of degree $n - 2$ with a_0 as its leading coefficient.

Continuing this process for a total of n times, we obtain n linear factors and a final quotient which is simply the leading coefficient a_0. Hence, we may write (1) in the form

$$f(x) \equiv a_0(x - r_1)(x - r_2) \cdots (x - r_n) = 0 \tag{2}$$

where $r_1, r_2, \cdots, r_n$ are n roots of equation (1).

We will now show that these are the only roots of (1). Suppose that r, a number different from any of these roots, is also a root of equation (1). Substituting this value in (2), we must have

$$f(x) \equiv a_0(r - r_1)(r - r_2) \cdots (r - r_n) = 0.$$

But this is impossible because all of the factors a_0, $r - r_1$, $r - r_2$, $\cdots$, $r - r_n$ are different from zero. Hence equation (1) has exactly n roots and the proof is complete.

NOTE. Any of the roots of equation (1) may be real or complex, and any of them may be repeated. A repeated root of multiplicity m is counted as m roots.

The factored form of the rational integral equation, as given by equation (2), suggests a direct method for forming an equation whose roots are given. This is illustrated in

Example 1. Form the rational integral equation having 1 and -3 as distinct roots and 2 as a double root.

SOLUTION. The left member of the required equation has the factors $x - 1$, $x + 3$, and $(x - 2)^2$. Hence the equation is given by

$$(x - 1)(x + 3)(x - 2)^2 = 0$$

or

$$x^4 - 2x^3 - 7x^2 + 20x - 12 = 0.$$

In connection with the proof of Theorem 4, for the first root r_1 of $f(x) = 0$, we wrote

$$f(x) \equiv (x - r_1)Q_1(x),$$

where $Q_1(x)$, a polynomial of degree $n - 1$, is the quotient obtained by dividing $f(x)$ by $x - r_1$. The equation $Q_1(x) = 0$ is then called a *depressed equation* with respect to $f(x) = 0$. When a root of a given equation is known, it is usually desirable to remove this root and obtain the depressed equation. The remaining roots should then be sought from the depressed equation rather than from the original equation since, in general, the lower the degree of an equation, the easier it is to solve. We illustrate the procedure in

Example 2. Show that 2 and -1 are roots of the equation $x^4 + x^3 - 2x^2 - 6x - 4 = 0$, and find the remaining roots.

SOLUTION. We first verify 2 as a root by synthetic division. Thus,

$$\begin{array}{l|l} 1+1-2-6-4 & \underline{2} \\ \underline{\quad +2+6+8+4} & \\ 1+3+4+2 & \underline{+0} \end{array}$$

The depressed equation is $x^3 + 3x^2 + 4x + 2 = 0$. We use this equation, rather than the original one, to verify -1 as a root. Thus, by synthetic division,

$$\begin{array}{l|l} 1+3+4+2 & \underline{-1} \\ \underline{\quad -1-2-2} & \\ 1+2+2 & \underline{+0} \end{array}$$

The depressed equation is now the quadratic equation $x^2 + 2x + 2 = 0$, the roots of which, by the quadratic formula, are readily found to be the conjugate complex numbers $-1 \pm i$.

From Theorem 4 it is possible to obtain an important result, which we state as

Theorem 5. *If two polynomials, each of degree not greater than n, are identically equal, then the coefficients of like powers of the variable are equal.*

PROOF. Let the two polynomials be represented by

$$P_1(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n,$$
$$P_2(x) = b_0x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n.$$

Since $P_1(x) \equiv P_2(x)$, it follows that

$$P_1(x) - P_2(x) \equiv 0,$$

or, in terms of their polynomial representations,

$$(3)\quad (a_0 - b_0)x^n + (a_1 - b_1)x^{n-1} + \cdots + (a_{n-1} - b_{n-1})x + a_n - b_n = 0.$$

Now, by Theorem 4, there are exactly n values of x for which the relation (3) holds. Hence, if relation (3) is to be an identity, that is, if it is to hold for *all* values of x, and therefore for more than n values of x, the coefficients in relation (3) must all vanish. In other words, we must have

$$a_0 - b_0 = 0,\quad a_1 - b_1 = 0, \cdots,\quad a_n - b_n = 0,$$

whence $$a_0 = b_0,\quad a_1 = b_1, \cdots,\quad a_n = b_n.$$

This completes the proof.

Corollary. *If two polynomials, each of degree not greater than n, are equal to each other for more than n distinct values of the variable, the coefficients of like powers are equal, and the equality is an identity.*

As an illustration of Theorem 5, we have

Example 3. Find the values of A, B, and C if the following identity is to hold:

$$2x^2 - 3x - 11 \equiv A(x^2 - 1) + B(x^2 + 3x + 2) + C(x^2 + x - 2).$$

SOLUTION. If we collect the various powers of x in the right member of the given identity, we have

$$2x^2 - 3x - 11 \equiv (A + B + C)x^2 + (3B + C)x - A + 2B - 2C.$$

In accordance with Theorem 5, for this identity to hold, the coefficients of like powers of x must be equal. Hence we must have

$$A + B + C = 2, \quad 3B + C = -3, \quad -A + 2B - 2C = -11.$$

The solution of this system of 3 equations in 3 unknowns gives us $A = 1$, $B = -2$, and $C = 3$, the required values.

EXERCISES. GROUP 38

In each of Exs. 1–12, form the rational integral equation having the given roots.

1. $1, -1, 2$.
2. $\frac{1}{2}, 2, -3$.
3. $2, -2, 4, -3$.
4. $5, 1 \pm \sqrt{2}$.
5. $1, 1 \pm \sqrt{3}$.
6. $4, \dfrac{1 \pm 2\sqrt{2}}{2}$.
7. $1 \pm \sqrt{2}, 2 \pm \sqrt{3}$.
8. $1, 1, -2, -2$.
9. $2, -3, 1, 1, 1$.
10. $1, 4, 1 \pm i$.
11. $2, -5, 1 \pm 2i$.
12. $\dfrac{1 \pm i}{2}, 3 \pm 2i$.

In each of Exs. 13–20, show that the given equation has as root(s) the indicated value(s) of r, and find the remaining roots.

13. $x^3 - 7x - 6 = 0, r = 3$.
14. $3x^3 - x^2 - 3x + 1 = 0, r = \frac{1}{3}$.
15. $x^3 - 6x^2 + 13x - 10 = 0, r = 2$.
16. $6x^4 - 41x^3 + 64x^2 + 19x - 12 = 0, r = 4, -\frac{1}{2}$.
17. $x^4 - x^3 - 9x^2 + 3x + 18 = 0, r = 3, -2$.
18. $2x^4 - 3x^3 - 14x^2 + 2x + 4 = 0, r = -2, -\frac{1}{2}$.
19. $x^4 + 4x^3 - x^2 + 16x - 20 = 0, r = 1, -5$.
20. $3x^4 + 11x^3 - 34x^2 + 46x - 12 = 0, r = \frac{1}{3}, -6$.
21. Show that the equation $x^4 - 11x^2 - 12x + 4 = 0$ has -2 as a double root, and find the remaining roots.
22. Show that the equation $8x^5 - 44x^4 + 94x^3 - 85x^2 + 34x - 5 = 0$ has $\frac{1}{2}$ as a triple root, and find the remaining roots.

In each of Exs. 23–25, find the values of A, B, and C if the given identity is to hold.

23. $5x + 1 \equiv A(x + 2) + B(x - 1)$.

24. $7x^2 + 5x - 8 \equiv A(x^2 + x - 6) + B(x^2 + 4x + 3) + C(x^2 - x - 2)$.

25. $-x - 2 \equiv A(x^2 + x + 1) + (Bx + C)(x + 1)$.

11.7. NATURE OF THE ROOTS

In this section we continue our efforts to narrow the field of search for the roots of the rational integral equation $f(x) = 0$. In particular, we consider several theorems whereby it is possible to obtain some information concerning the nature of the roots before actually solving the equation. Thus our first theorem is concerned with the occurrence of complex roots.

Theorem 6. *If any complex number $a + bi$ is a root of the rational integral equation $f(x) = 0$ with real coefficients, then the conjugate complex number $a - bi$ is also a root.*

PROOF. Let us substitute $a + bi$ for x in the given rational integral equation

(1) $$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0.$$

In the evaluation of the left member, even powers of bi will give real numbers, while odd powers of bi will give various multiples of the imaginary unit i. Let the algebraic sum of all real numbers resulting from this substitution be represented by the real number A, and let the algebraic sum of all the imaginary numbers be represented by Bi, where B is a real number. Then, since $a + bi$ is a root of (1), we have

(2) $$A + Bi = 0,$$

whence, by the definition of a zero complex number (Sec. 8.2),

(3) $$A = 0 \quad \text{and} \quad B = 0.$$

Now if we substitute $a - bi$ in the left member of (1), the even powers of $-bi$ will be the same as the even powers of bi, while the odd powers of $-bi$ will differ from the odd powers of bi only in sign. Hence, if the real numbers A and B have the same significance as above, the result of this substitution is $A - Bi$ for which, in view of (3), we may write

$$A - Bi = 0.$$

Hence, $a - bi$ is a root of equation (1), and the theorem is established.

As an immediate consequence of this theorem, we have

Corollary 1. *A rational integral equation with real coefficients and of odd degree must have at least one real root.*

We may also obtain another very important result from Theorem 6. Let a pair of conjugate complex roots of equation (1) be represented by $a \pm bi$. Then, by the factor theorem (Sec. 11.3), $x - (a + bi)$ and $x - (a - bi)$ are both factors of the polynomial $f(x)$. Hence, their product $(x - a - bi)(x - a + bi) = x^2 - 2ax + a^2 + b^2$ is also a factor of $f(x)$. Also, for each real root r (rational or irrational) of equation (1), there corresponds a linear factor $x - r$ of $f(x)$. Combining these facts, we have

Corollary 2. *Any polynomial in the single variable x and with real coefficients may be expressed as the product of linear and quadratic factors with real coefficients, each linear factor corresponding to a real zero and each quadratic factor to a pair of conjugate complex zeros.*

As an illustration of Theorem 6, we have the following

Example. Given that $1 + 2i$ is a root of the equation

(4) $$x^4 - 5x^3 + 7x^2 - 7x - 20 = 0,$$

find the remaining roots.

SOLUTION. By Theorem 6, the conjugate complex number $1 - 2i$ is also a root of (4). Hence $(x - 1 - 2i)(x - 1 + 2i) = x^2 - 2x + 5$ is a factor of the left member of (4). By division, the other factor is found to be $x^2 - 3x - 4$. This gives us the depressed equation $x^2 - 3x - 4 = 0$ whose roots are readily found to be -1 and 4. Hence the required roots are $1 - 2i$, -1, and 4.

There is a theorem on irrational roots analogous to Theorem 6. Let a and b be two rational numbers and let $\sqrt{b}$ be an irrational number. Then $a + \sqrt{b}$ is called a *quadratic surd* and $a - \sqrt{b}$ is called its *conjugate quadratic surd*. (See Note, Sec. 5.5.) By a method similar to that employed in proving Theorem 6, we may establish

Theorem 7. *If any quadratic surd $a + \sqrt{b}$ is a root of the rational integral equation $f(x) = 0$ with rational coefficients, then the conjugate quadratic surd $a - \sqrt{b}$ is also a root.*

NOTE. At the close of Sec. 2.8 on number fields, it was stated that a property or theorem which is true in one field may not be true in another field. Theorems 6 and 7 are examples of this fact. Thus, with reference to Theorem 6, if $a + bi$ is a root of a rational integral equation whose coefficients are not all real numbers, it does not necessarily follow that the conjugate $a - bi$ is also a root.

EXERCISES. GROUP 39

In each of Exs. 1–12, given the indicated root(s) of the equation, find the remaining roots.

1. $x^3 + x^2 - 4x + 6 = 0;\ 1 - i.$
2. $x^3 - 4x^2 + 14x - 20 = 0;\ 1 + 3i.$
3. $x^4 - 6x^3 + 14x^2 - 14x + 5 = 0;\ 2 - i.$
4. $x^4 + x^3 + x^2 + 11x + 10 = 0;\ 1 + 2i.$
5. $x^3 + x^2 - 5x - 5 = 0;\ \sqrt{5}.$
6. $x^3 - 6x^2 + 7x + 4 = 0;\ 1 - \sqrt{2}.$
7. $x^4 - 9x^3 + 27x^2 - 33x + 14 = 0;\ 3 + \sqrt{2}.$
8. $x^4 - 3x^3 - 6x^2 + 14x + 12 = 0;\ 1 - \sqrt{3}.$
9. $x^5 - 7x^4 + 16x^3 - 32x^2 + 15x - 25 = 0;\ i, 1 - 2i.$
10. $x^5 - x^4 - 5x^3 + x^2 + 6x + 2 = 0;\ \sqrt{2}, 1 + \sqrt{2}.$
11. $x^5 - 8x^4 + 26x^3 - 40x^2 + 16x = 0;\ 2 + \sqrt{2}, 2 + 2i.$
12. $x^6 - 2x^5 - 4x^4 - 8x^3 - 77x^2 + 90x + 360 = 0;\ \sqrt{5}, 3i.$

In each of Exs. 13–15, form the equation of lowest degree with real coefficients having the indicated roots.

13. $-2, 3 + i.$ 14. $1, 3, 1 + 2i.$ 15. $2 + 4i, 2i.$

In each of Exs. 16–18, form the equation of lowest degree with rational coefficients having the indicated roots.

16. $1, 1 + \sqrt{5}.$ 17. $2, -3, 2 - \sqrt{3}.$ 18. $\sqrt{7}, 1 - \sqrt{2}.$

In each of Exs. 19–21, express the given polynomial as the product of linear and quadratic factors with real coefficients.

19. $x^3 + 3x^2 - 3x - 14.$ 20. $x^4 + 2x^3 + x^2 + 8x - 12.$
21. $x^4 - 2x^3 - 6x^2 - 7x - 4.$

In each of Exs. 22 and 23, express the given polynomial as the product of linear and quadratic factors with rational coefficients.

22. $x^4 - x^3 - 9x^2 + 3x + 18.$ 23. $2x^4 - 9x^3 + 10x^2 + x - 2.$
24. Establish Corollary 1 of Theorem 6 (Sec. 11.6).
25. Establish Theorem 7 (Sec. 11.6).

11.8. DESCARTES' RULE OF SIGNS

We now continue our study of the nature of the roots of a rational integral equation by considering a very important theorem known as *Descartes' rule of signs.* By means of this rule, it is possible to determine the maximum number of positive and negative roots of a rational integral equation with real coefficients. However, before stating and discussing this theorem, it will be necessary to establish certain preliminary facts.

We first consider the determination of any possible zero roots of a rational integral equation, for such roots are neither positive nor negative. It is clear that if the equation lacks the constant term, but not a first degree term, it has a single zero root; if it lacks both constant and first degree terms, but not a second degree term, it has two zero roots, and so on. In general, if the equation has the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-r}x^r = 0, \quad a_0 \neq 0,$$

where the term of lowest degree is $a_{n-r}x^r$, then the equation has exactly r zero roots. In such a case we remove these r zero roots by factoring out x^r and then continue to work with the depressed equation of degree $n - r$. It will be understood hereafter that the first step in the solution of a rational integral equation is the removal of any zero roots. We may note here that an equation in which all powers of x and the constant term are present is said to be *complete*, otherwise, *incomplete*.

Let $f(x) = 0$ represent a rational integral equation. If x is replaced by $-x$ throughout this equation, we obtain another equation $f(-x) = 0$ whose roots are the roots of $f(x) = 0$ with their signs changed. For, if $x = r$ is a root of $f(x) = 0$, then $-x = r$ or $x = -r$ is a root of $f(-x) = 0$. Also, if $-x$ is substituted for x in the polynomial $f(x)$, the new polynomial $f(-x)$ differs from $f(x)$ only in the signs of the terms of odd degree, the constant, if any, being considered of even (zero) degree. For example, consider the equation

$$x^4 + 2x^3 - 13x^2 - 14x + 24 = 0. \tag{1}$$

Then the equation whose roots are numerically equal but opposite in sign to those of equation (1) is

$$x^4 - 2x^3 - 13x^2 + 14x + 24 = 0. \tag{2}$$

The student may readily verify that 1, 3, -2, -4 are the roots of (1) and that -1, -3, 2, 4 are the roots of (2). Summarizing, we may state that in order to transform a given equation into another whose roots are opposite in sign, we need merely change the signs of the terms of odd degree. We shall see later (Sec. 11.11) that this is a special case of a more general transformation.

Let $f(x)$ represent a polynomial in x with real coefficients and arranged in descending powers of x. If two successive terms differ in sign, there is said to be a *variation in sign*. Thus, in the left member of equation (1) above, there are two variations in sign, one from $2x^3$ to $-13x^2$ and the other from $-14x$ to 24. Note that a variation in sign occurs for two successive terms even though some intermediate powers may be missing. Thus, in the polynomial $x^7 - 2x^4 + 3x^3 - 2$, there is only one variation in sign from x^7 to $-2x^4$, one from $-2x^4$ to $3x^3$, and only one from $3x^3$ to -2. Our purpose in introducing the term variation in sign at this time is that it is a basic concept in Descartes' rule of signs which we will now state in full as

Theorem 8. The proof of this theorem is omitted; it is beyond the scope of this book.

Theorem 8. (*Descartes' Rule of Signs*). *Let $f(x) = 0$ be a rational integral equation with real coefficients and no zero roots. Then*

1. The number of positive roots of $f(x) = 0$ is either exactly equal to the number of variations in sign in $f(x)$ or else is less than that number by a positive even integer.

2. The number of negative roots of $f(x) = 0$ is either exactly equal to the number of variations in sign in $f(-x)$ or else is less than that number by a positive even integer.

NOTES. 1. Part 2 of this theorem concerning negative roots is an immediate consequence of Part 1, since the positive roots of $f(-x) = 0$ are the negative roots of $f(x) = 0$.

2. This theorem also gives us some information concerning the possible number of complex roots. If $f(x) = 0$ is of degree n, it has exactly n roots (Theorem 4, Sec. 11.6). Hence the number of complex roots is equal to n diminished by the sum of the positive and negative roots.

As our first illustration of this theorem, consider equation (1) above where, as we have previously noted, there are two variations in sign. Hence this equation has either exactly two positive roots or none. Furthermore, since equation (2), whose roots are opposite in sign to those of equation (1), has two variations in sign, equation (1) has either exactly two negative roots or none. Also, since equation (1) is of degree 4, it may have four, two, or no complex roots. Thus there are four possible combinations for the roots of equation (1), as shown in the following table.

Positive	Negative	Complex
2	2	0
2	0	2
0	2	2
0	0	4

In this particular case we happen to know by a previous check that there are exactly 2 positive and 2 negative roots.

Under certain conditions Descartes' rule gives very precise information. For example, if $f(x)$ has only 1 variation in sign, then $f(x) = 0$ has exactly 1 positive root, for we cannot diminish 1 by a positive even integer. Similarly, if $f(x) = 0$ has an odd number of variations in sign, then $f(x) = 0$ has at least 1 positive root. Similar remarks apply for negative roots.

We note in the table above that there are either no complex roots or else an even number of them. This must necessarily be the case because complex roots occur in conjugate pairs (Theorem 6, Sec. 11.7).

As a further illustration of Descartes' rule we have the following

Example. By means of Descartes' rule of signs, find all possible information about the nature of the roots of the given equation.

(a) $x^5 + 3x^4 + 2x^3 - x^2 - 3x - 2 = 0.$

(b) $x^6 - 3x^5 + x^4 - x^3 - 6x^2 = 0.$

SOLUTION. (a) We first write

$$f(x) = x^5 + 3x^4 + 2x^3 - x^2 - 3x - 2$$

whence $$f(-x) = -x^5 + 3x^4 - 2x^3 - x^2 + 3x - 2.$$

$f(x)$ has only 1 variation in sign. Hence there is exactly 1 positive root.

$f(-x)$ has 4 variations in sign. Hence there are either 4, 2 or no negative roots.

Therefore the possible combinations of positive, negative, and complex roots are as shown in the accompanying table where the number of complex roots is given in the third column under "c."

+	−	c
1	4	0
1	2	2
1	0	4

(b) By inspecting we see that the given equation has two zero roots. Factoring out x^2, we have

$$f(x) = x^4 - 3x^3 + x^2 - x - 6$$

whence $$f(-x) = x^4 + 3x^3 + x^2 + x - 6.$$

$f(x)$ has 3 variations in sign. Hence there are either 3 positive roots or 1 positive root.

$f(-x)$ has 1 variation in sign. Hence there is exactly 1 negative root.

The possible combinations of zero, positive, negative, and complex roots are as shown in the accompanying table.

0	+	−	c
2	3	1	0
2	1	1	2

EXERCISES. GROUP 40

In each of Exs. 1–16, by means of Descartes' rule, find all possible information about the nature of the roots of the given equation.

1. $2x^4 + x^2 + 2x - 3 = 0.$
2. $x^5 - 4x^4 + 3x^3 - 5 = 0.$
3. $3x^3 + 9x^2 - 7x + 4 = 0.$
4. $x^4 + 2x^3 - 3x^2 + 2x + 2 = 0.$
5. $2x^6 + 3x^4 + 2x^2 + 9 = 0.$
6. $x^7 + 5x^4 + 2x^3 + 7x + 1 = 0.$
7. $x^5 + 3x^3 + 5x = 0.$
8. $4x^4 - 3x^3 + 2x^2 - x + 2 = 0.$
9. $x^8 - 1 = 0.$
10. $x^7 - 1 = 0.$
11. $x^8 + 1 = 0.$
12. $x^7 + 1 = 0.$
13. $x^5 - 2x^4 + 5x^3 - 7x^2 = 0.$
14. $x^7 + 2x^5 - 3x^4 + 8x^3 - 9x = 0.$

15. $x^9 + 4x^7 - 6x^6 + 4x^4 - 8 = 0$. 16. $2x^8 - 3x^6 + 9x^3 - x^2 + 5 = 0$.

17. Show that the equation $3x^5 - x^4 + 2x - 8 = 0$ has at least two complex roots.

18. Show that the equation $x^7 + 4x^6 + 2x^3 + 9x^2 + 6 = 0$ has at least four complex roots.

19. Show that the equation $4x^4 - 3x^3 - x - 10 = 0$ has exactly two complex roots.

20. Show that the equation $2x^6 + 3x^4 - 2x^2 - 6 = 0$ has exactly four complex roots.

21. For the equation $x^n - 1 = 0$, show that (a) if n is even, there are exactly two real roots ± 1 and $n - 2$ complex roots; (b) if n is odd, there are exactly one real root $+1$ and $n - 1$ complex roots.

22. For the equation $x^n + 1 = 0$, show that (a) if n is even, all n roots are complex; (b) if n is odd, there are exactly one negative root -1 and $n - 1$ complex roots.

23. Show that a given equation may be transformed into another whose roots are opposite in sign by changing the signs of the terms of even degree, the constant term being considered of even degree.

In the following exercises, all equations are rational integral with real coefficients.

24. Show that an equation whose terms are all positive has no positive roots.

25. Show that an equation whose terms of even power are all of one sign and whose terms of odd power are all of the opposite sign has no negative roots.

26. Show that a complete equation whose terms are alternately positive and negative has no negative roots.

27. Show that an equation having only odd powers (and no constant term), and these all of the same sign, has no real root except zero.

28. Show that an equation having only even powers (and a constant term), and these all of the same sign, has no real roots.

29. If all the roots of a complete equaiton $f(x) = 0$ are real, show that the number of positive roots is exactly equal to the number of variations in sign of $f(x)$, and that the number of negative roots is exactly equal to the number of variations in sign of $f(-x)$.

30. If the equation $f(x) = 0$ has no zero roots, show that it has at least as many complex roots as the difference between the degree of the equation and the total number of variations in sign of $f(x)$ and $f(-x)$.

11.9. RATIONAL ROOTS

We next consider the determination of any rational roots which a given rational integral equation may possess. For this purpose we have the following very useful and important theorem.

Theorem 9. *Let the rational fraction p/q, reduced to its lowest terms, be a root of the rational integral equation*

$$(1)\qquad a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$$

whose coefficients are integers (or zero) but $a_0 \neq 0$ and $a_n \neq 0$. Then p is an exact divisor of a_n and q is an exact divisor of a_0.

PROOF. Since p/q is a root of equation (1), we have

$$(2)\qquad a_0\left(\frac{p}{q}\right)^n + a_1\left(\frac{p}{q}\right)^{n-1} + \cdots + a_{n-1}\left(\frac{p}{q}\right) + a_n = 0.$$

Multiplying both sides of (2) by q^n, we have

$$(3)\qquad a_0p^n + a_1p^{n-1}q + \cdots + a_{n-1}pq^{n-1} + a_nq^n = 0.$$

Transposing a_nq^n to the right side of (3) and then factoring out p from the left side, we obtain

$$(4)\qquad p(a_0p^{n-1} + a_1p^{n-2}q + \cdots + a_{n-1}q^{n-1}) = -a_nq^n.$$

Since $p, q, a_0, a_1, \cdots, a_n$ are all integers, it follows that both sides of (4) are integers. Since p is a factor of the left member, it must also be a factor of the right member. But p and q have no factor in common (except ± 1); hence p is an exact divisor of a_n.

From equation (3), we have

$$(5)\qquad q(a_1p^{n-1} + \cdots + a_{n-1}pq^{n-2} + a_nq^{n-1}) = -a_0p^n.$$

If the same reasoning is applied to equation (5) that we applied to equation (4), we find that q is an exact divisor of a_0.

From this theorem we have the important

Corollary. *In the rational integral equation* (1), *whose coefficients are integers, if the leading coefficient $a_0 = 1$ and the constant term $a_n \neq 0$, then any rational root is an integer and an exact divisor of a_n.*

NOTE. We can also make use of Theorem 9 when the coefficients are rational but not integers. In such a case we need merely multiply by the least common denominator of these coefficients, thus obtaining an equivalent equation with integral coefficients to which Theorem 9 applies.

It should be particularly observed that the importance of Theorem 9 lies in the fact that it restricts our search for rational roots to a limited number of possibilities. This will be seen in the following examples.

Example 1. Find all the roots of the equation

$$2x^4 - x^3 - 4x^2 + 10x - 4 = 0. \tag{6}$$

SOLUTION. We first apply Descartes' rule of signs to equation (6) and thus obtain the results shown in the accompanying table.

+	−	c
3	1	0
1	1	2

We next apply Theorem 9 for any possible rational roots p/q by writing out the factors of the constant term -4 for values of p and the factors of the leading coefficient 2 for values of q. This is conveniently exhibited thus:

$$\frac{p = \pm 1, \pm 2, \pm 4}{q = \pm 1, \pm 2}.$$

Hence we have the following eight possible rational roots: $\pm 1, \pm\frac{1}{2}, \pm 2, \pm 4$. In testing these values we find that $\frac{1}{2}$ and -2 are roots. As soon as one negative root is found, there is no need for testing further for negative roots, in accordance with the findings of Descartes' rule of signs. In all cases, as soon as a root is found it should be removed and the testing continued with the depressed equation. This may be done conveniently by synthetic division as here shown.

$$\begin{array}{lr}
2 - 1 - 4 + 10 - 4 & \underline{|\tfrac{1}{2}} \\
\underline{\quad + 1 + 0 - \ 2 + 4} & \\
2 + 0 - 4 + \ 8 & \underline{|-2} \\
\underline{\quad - 4 + 8 - \ 8} & \\
2 - 4 + 4 &
\end{array}$$

The final depressed equation is $2x^2 - 4x + 4 = 0$ or $x^2 - 2x + 2 = 0$ whose roots, by the quadratic formula, are readily found to be $1 \pm i$. Hence the roots of the given equation (6) are $\frac{1}{2}, -2, 1 \pm i$.

At this point we have gone as far as we intend to go in finding rational roots. It is therefore an appropriate place to summarize the various steps to be taken in obtaining such roots, as shown in the

Procedure for Rational Roots

In order to obtain the rational roots of a given rational integral equation with rational coefficients, the following steps should be taken in order:

1. If there are any zero roots, they should be removed, and the resulting depressed equation only should be considered in the steps following.
2. Apply Descartes' rule of signs (Theorem 8, Sec. 11.8) to determine the possible nature and distribution of the roots. Use this information as a guide for any testing in the succeeding steps.

3. Apply Theorem 9 and its corollary (Sec. 11.9) for determining any possible rational roots. Test for these roots, and as each root is found, remove it and continue with the depressed equation.

4. After all rational roots have been removed in Step 3, the depressed equation, if any, has only irrational and (or) complex roots. If this depressed equation is quadratic, it may be solved for the remaining roots.

As an illustration of the above procedure we have

Example 2. Find all the roots of the equation

$$x^6 + 3x^5 - 13x^4 - 25x^3 + 50x^2 + 24x = 0. \tag{7}$$

SOLUTION. 1. By inspection, we see that equation (7) has one zero root. Removing this root we have the depressed equation

$$x^5 + 3x^4 - 13x^3 - 25x^2 + 50x + 24 = 0. \tag{8}$$

2. Applying Descartes' rule to equation (8), we obtain the results shown in the accompanying table.

+	−	c
2	3	0
2	1	2
0	3	2
0	1	4

3. Since the leading coefficient of equation (8) is unity and the coefficients are all integers, it follows from the corollary to Theorem 9 (Sec. 11.9) that any rational root must be an integer and an exact divisor of the constant term 24. Hence possible rational roots are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$. By actual test, we find that 2, −3, and −4 are roots. The removal of these roots leading to the depressed equation is shown below.

$$\begin{array}{rrrrrrl}
1 & +3 & -13 & -25 & +50 & +24 & \underline{|2} \\
 & +2 & +10 & -6 & -62 & -24 & \\ \hline
1 & +5 & -3 & -31 & -12 & & \underline{|-3} \\
 & -3 & -6 & +27 & +12 & & \\ \hline
1 & +2 & -9 & -4 & & & \underline{|-4} \\
 & -4 & +8 & +4 & & & \\ \hline
1 & -2 & -1 & & & &
\end{array}$$

4. In Step 3 we actually found 1 positive and 2 negative roots. But, in view of the results from Descartes' rule (Step 2), there must be another positive root and another negative root. These two roots must therefore be irrational. We verify this conclusion by solving the depressed equation $x^2 - 2x - 1 = 0$ whose roots are found to be $1 \pm \sqrt{2}$. Hence the roots of the given equation are $0, 2, -3, -4, 1 \pm \sqrt{2}$.

EXERCISES. GROUP 41

1. Establish the corollary to Theorem 9 (Sec. 11.9).

In each of Exs. 2–17, find all the roots of the given equation.

2. $2x^3 - 9x^2 + 12x - 4 = 0.$
3. $3x^3 - 4x^2 - 35x + 12 = 0.$
4. $4x^4 - 39x^3 + 54x^2 + 16x = 0.$
5. $2x^3 + \frac{29}{3}x^2 - \frac{40}{3}x + 4 = 0.$
6. $2x^4 + 3x^3 - 10x^2 - 12x + 8 = 0.$
7. $9x^4 + 15x^3 - 143x^2 + 41x + 30 = 0.$
8. $4x^4 + 2x^3 - 8x^2 - 3x + 3 = 0.$
9. $4x^5 - 4x^4 - 5x^3 + x^2 + x = 0.$
10. $3x^5 + 5x^4 + x^3 + 5x^2 - 2x = 0.$
11. $x^6 - x^5 - 2x^3 - 4x^2 = 0.$
12. $2x^4 - 3x^3 - x^2 - 12x - 36 = 0.$
13. $3x^4 - 4x^3 + 28x^2 - 36x + 9 = 0.$
14. $12x^5 + 4x^4 + 7x^3 + 14x^2 - 34x + 12 = 0.$
15. $6x^4 + 11x^3 - 8x^2 + 37x - 6 = 0.$
16. $8x^4 + 10x^3 + 9x^2 + x - 1 = 0.$
17. $8x^4 - 28x^3 + 34x^2 - 175x - 100 = 0.$

In each of Exs. 18–23, find the rational roots of the given equation.

18. $3x^3 + 11x^2 + 8x - 4 = 0.$
19. $x^5 + 3x^4 + 5x^3 + 8x^2 + 6x + 4 = 0.$
20. $2x^6 + x^5 - 2x^4 - x^3 - 12x^2 - 6x = 0.$
21. $x^7 - 3x^6 + x^5 - 3x^4 + x^3 - 3x^2 + x - 3 = 0.$
22. $12x^6 - 13x^5 - 12x^4 + 26x^3 - 25x^2 + 2 = 0.$
23. $3x^8 + x^7 + x^6 + x^5 + 10x^4 + 4x^3 + 4x^2 + 4x - 8 = 0.$

In each of Exs. 24–27, show that the given equation has no rational roots.

24. $x^4 + 4x^2 - x + 6 = 0.$
25. $x^4 + 2x^3 - 3x^2 - 4x + 3 = 0.$
26. $2x^4 - x^3 + 4x^2 + x + 2 = 0.$
27. $x^5 - 4x^4 + x^3 + 2x^2 - 8x + 2 = 0.$

28. The dimensions of a rectangular box are 3 ft, 5 ft, and 7 ft. If each dimension is increased by the same amount, the volume is tripled. Find the increase in each dimension.

29. The dimensions of a rectangular box are 6 ft, 8 ft, and 12 ft. If each dimension is decreased by the same amount, the volume is decreased by 441 cu ft. Find the decrease in each dimension.

30. Equal squares are cut from the corners of a rectangular piece of card board 7 in. long and 6 in. wide, and the remaining rectangular portions along the sides are folded up to form an open box whose volume is 15 cu in. Find the side of each square cut out. (Two solutions.)

11.10. IRRATIONAL ROOTS

If a rational integral equation has any irrational roots, they may be determined by various methods. Two of these methods will be considered in this chapter, one in the present section and the other in Sec. 11.12.

For a rational integral equation with rational coefficients, we will first follow the procedure for rational roots as outlined in Sec. 11.9. Thus any zero and (or) rational roots are first removed, and any irrational roots are obtained from the depressed equation. If the depressed equation is a quadratic, the roots are readily obtained by the quadratic formula. Hence, in the following discussion we will assume that the depressed equation is of degree 3 or higher. In this case irrational roots are generally determined in decimal form, and the degree of accuracy depends upon the number of decimal places obtained. The process, therefore, is essentially one of *approximation.*

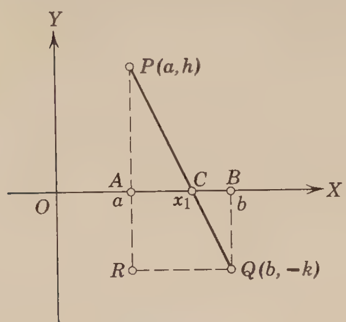


Figure 39

The method of approximation discussed in this section is called *linear interpolation.* It is based on the assumption that a small segment of a continuous graph may be considered a straight line without introducing an appreciable error. This is, of course, only an approximation but becomes better as the length of the segment is diminished.

To describe the method of linear interpolation, let us consider the graph of the polynomial function $f(x)$ with real coefficients. Let a and b be two positive numbers very close in value and such that $b > a$. For $x = a$, suppose that $f(a) = h > 0$, and for $x = b$, suppose that $f(b) = -k < 0$. Then $f(x)$ has a zero between a and b (Sec. 11.5). This situation is represented graphically in Fig. 39 where $P(a, h)$ and $Q(b, -k)$ are two neighboring points on the graph. The points A and B, respectively, are the feet of the perpendiculars dropped from P and Q to the X-axis. Let R be the point of intersection of the extension of PA and the line through Q parallel to the X-axis. We assume that the graph of $f(x)$ between P and Q is a straight line intersecting the X-axis in the point C between A and B. Then the abscissa x_1 of the point C is the approximate value of the zero of $f(x)$ which lies between a and b. This value of x_1 may now be readily computed. From the similar right triangles PAC and PRQ, we have the relation

(1) $$\frac{AC}{RQ} = \frac{AP}{RP}.$$

Now $RQ = AB = b - a$, $AP = h$, and $RP = h + k$. Substituting these values in (1), we have

$$\frac{AC}{b-a} = \frac{h}{h+k}, \quad \text{whence } AC = \frac{h(b-a)}{h+k}.$$

Since a, b, h, and k are all known quantities, AC is readily determined. Adding its value to a, we obtain the required value of x_1, the *first* approximation.

Starting with this first approximation, we may repeat the process to obtain a second and more accurate approximation. The process may be repeated successively as often as required in order to obtain any desired degree of accuracy.

To illustrate linear interpolation in a specific case, we have the following

Example. Show that the equation

$$f(x) = x^3 - 5x^2 + 2x + 6 = 0 \tag{2}$$

has a root between 1 and 2, and find it correct to one decimal place.

SOLUTION. By synthetic division we find $f(1) = 4$ and $f(2) = -2$ so that equation (2) definitely has a root between 1 and 2. We next plot this information as shown in Fig. 40(*a*), the lettering being the same as in Fig. 39. Then from relation (1) above we have

$$\frac{AC}{1} = \frac{4}{6}, \quad \text{whence } AC = \frac{2}{3} = 0.6^+.$$

Our first approximation is therefore $x_1 = 1 + 0.6 = 1.6$.

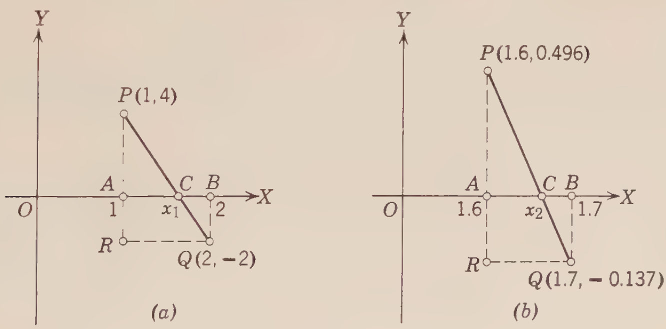


Figure 40

To ensure the accuracy of the required root to the first decimal place, we repeat the process to obtain the second decimal place. Thus, we find $f(1.6) = 0.496$ and $f(1.7) = -0.137$, so that equation (1) has a root between 1.6 and 1.7. This information is plotted in Fig. 40(*b*), where the same lettering is used again. Here $RQ = 0.1$, $AP = 0.496$, and $RP = 0.137 + 0.496 = 0.633$. Hence, by relation (1), we have

$$\frac{AC}{0.1} = \frac{0.496}{0.633}, \quad \text{whence } AC = \frac{0.0496}{0.633} = 0.07^{+}.$$

Thus our second approximation is $x_2 = 1.6 + 0.07 = 1.67$.

The required root, correct to one decimal place, is therefore 1.7.

NOTES. 1. Care should be taken to test each approximation to make sure that a root lies between two consecutive values. This is particularly important for the first approximation since we here consider the segment of the graph which is maximum in length and where, therefore, the approximation is least accurate. For example, the first approximation in a particular case may indicate a root between 1.6 and 1.7, but testing may show that the root actually lies between, say, 1.2 and 1.3.

2. While the method of linear interpolation becomes more and more accurate with successive approximations, the amount of computation also increases considerably. The method, however, has the distinct advantage that it may be used also to approximate the irrational roots of nonalgebraic equations, that is, of transcendental equations such as trigonometric and logarithmic equations. The labor of computation may be reduced to some extent by the use of tables of functions and calculating machines.

11.11. TRANSFORMATION OF EQUATIONS

By a *transformation* we mean an operation whereby one relation or expression is changed into another in accordance with a given law. Generally the purpose of a transformation is to change a given relation into a more useful form. In particular, this section will be devoted to two types of transformations whereby a given rational integral equation is changed into another whose roots bear a specified relation to those of the original equation. These transformations are discussed at this time in anticipation of the work of the next section.

Theorem 10. *By multiplying the successive coefficients, starting with the second term, of the rational integral equation*

$$(1) \qquad a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$$

by $m, m^2, m^3, \cdots, m^n$, *we transform* (1) *into another equation of the form*

$$(2) \quad a_0y^n + ma_1y^{n-1} + m^2a_2y^{n-2} + \cdots + m^{n-1}a_{n-1}y + m^na_n = 0,$$

each of whose roots is m *times the corresponding root of equation* (1).

PROOF. Each root y of the transformed equation (2) is to correspond to each root x of the given equation (1) in accordance with the relation $y = mx$ from which $x = y/m$. Substituting this value of x in (1) we have

$$a_0\left(\frac{y}{m}\right)^n + a_1\left(\frac{y}{m}\right)^{n-1} + a_2\left(\frac{y}{m}\right)^{n-2} + \cdots + a_{n-1}\left(\frac{y}{m}\right) + a_n = 0.$$

Multiplying through by m^n, we obtain the required equation (2).

Corollary. *For the particular case* $m = -1$, *the roots of equation* (2) *are numerically equal but opposite in sign to those of equation* (1).

NOTES. 1. In using the transformation of Theorem 10, missing powers of x must be taken into account. This may be done by considering such terms to have zero coefficients.

2. The corollary has already been used in connection with Descartes' rule of signs (Sec. 11.8).

As an illustration of Theorem 10, we have

Example 1. Transform the equation

$$x^4 - 5x^3 - x + 5 = 0 \tag{3}$$

into another equation each of whose roots is twice the corresponding root of equation (3).

SOLUTION. We note first that the term of second degree is missing in equation (3). Hence, by Theorem 10, the transformed equation is

$$y^4 - (2)\cdot 5y^3 - (2)^2\cdot 0y^2 - (2)^3y + (2)^4\cdot 5 = 0, \text{ or}$$

$$y^4 - 10y^3 - 8y + 80 = 0. \tag{4}$$

The student should verify this result by showing that the roots of (3) are 1, 5, $\dfrac{-1 \pm \sqrt{3}i}{2}$, and that the roots of (4) are 2, 10, $-1 \pm \sqrt{3}i$.

Theorem 10 may also be used to transform a given equation whose leading coefficient is different from unity into another equation whose leading coefficient is unity and whose other coefficients are integers. The transformed equation is then one to which the corollary of Theorem 9 (Sec. 11.9) applies. We illustrate the procedure in

Example 2. Transform the equation

$$8x^4 + 10x^3 + 9x^2 + x - 1 = 0 \tag{5}$$

into another equation whose roots are equal to those of equation (5), multiplied by the smallest number which will make the leading coefficient of the new equation unity and all of its other coefficients integers.

SOLUTION. Dividing (5) through by 8, we have

$$x^4 + \frac{5}{4}x^3 + \frac{9}{8}x^2 + \frac{1}{8}x - \frac{1}{8} = 0.$$

In order to obtain a new equation with leading coefficient unity and all other coefficients integers, the smallest number by which we can multiply the roots of this equation is 4. Hence, by Theorem 10, the required equation is

$$x^4 + (4)\frac{5}{4}x^3 + (4)^2\frac{9}{8}x^2 + (4)^3\frac{1}{8}x - (4)^4\frac{1}{8} = 0, \quad \text{or}$$

$$x^4 + 5x^3 + 18x^2 + 8x - 32 = 0. \tag{6}$$

By the corollary of Theorem 9 (Sec. 11.9), the rational roots of (6) are found to be the integers 1, −2. Hence the rational roots of equation (5) are $\frac{1}{4}$, $-\frac{1}{2}$.

We now consider the transformation which is of basic importance in the method of approximation discussed in the next section.

Theorem 11. *The rational integral equation*

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0 \tag{7}$$

is transformed into the equation

$$a_0y^n + R_1y^{n-1} + R_2y^{n-2} + \cdots + R_{n-1}y + R_n = 0 \tag{8}$$

each of whose roots is h less than the corresponding root of equation (7) *and where the coefficients* $R_1, R_2, \cdots, R_n$ *may be obtained as follows:*

Divide $f(x)$ *by* $x - h$, *and call the remainder* R_n. *Divide the quotient by* $x - h$, *and call the remainder* R_{n-1}. *Continue this process for a total of n divisions, the last remainder being* R_1.

PROOF. Each root y of the transformed equation (8) is to correspond to each root x of the given equation (7) in accordance with the relation $y = x - h$, whence $x = y + h$. Substituting this value of x in (7) we have

$$a_0(y + h)^n + a_1(y + h)^{n-1} + \cdots + a_{n-1}(y + h) + a_n = 0, \tag{9}$$

each of whose roots is h less than the corresponding root of equation (7). To reduce equation (9) to a rational integral equation in y, we may expand the binomial powers, collect terms, and write the result in the form

$$a_0y^n + A_1y^{n-1} + A_2y^{n-2} + \cdots + A_{n-1}y + A_n = 0. \tag{10}$$

But we may determine the coefficients $A_1, A_2, \cdots, A_n$ much more simply. Thus, substituting $x - h$ for y in (10), we have

$$
\begin{aligned}
(11) \qquad a_0(x-h)^n + A_1(x-h)^{n-1} + A_2(x-h)^{n-2} \\
+ \cdots + A_{n-1}(x-h) + A_n = 0.
\end{aligned}
$$

If we divide the left side of (11) by $x - h$, we obtain a quotient and a remainder A_n. If we divide this quotient by $x - h$, we obtain another quotient and a remainder A_{n-1}. Continuing this process for a total of n divisions, we obtain A_1 as a final remainder. Designating the remainders $A_n, A_{n-1}, \cdots, A_1$ by $R_n, R_{n-1}, \cdots, R_n$, and substituting these values in equation (10), we obtain the required equation (8).

NOTES. 3. The divisions required by this theorem are conveniently performed by synthetic division, as illustrated in Example 3 below.

4. It follows from this theorem that if we wish to transform a given equation into another equation each of whose roots is h *more* than the corresponding root of the given equation, we diminish the roots of the given equation by $-h$.

Example 3. Transform the equation

$$(12) \qquad x^3 - 6x^2 + 5x + 12 = 0$$

into another equation, each of whose roots is two less than the corresponding root of equation (12).

SOLUTION. In accordance with Theorem 11, we divide successively by $x - 2$. With synthetic division, the work appears as follows:

$$
\begin{array}{llll}
1 - 6 + 5 + 12 & \underline{2} & & \\
\quad + 2 - 8 - \ 6 & & & \\
\hline
1 - 4 - 3 \mid +6 & & R_3 = 6 & \\
\quad + 2 - 4 \mid & & & \\
\hline
1 - 2 \mid -7 & & R_2 = -7 & \\
\quad + 2 \mid & & & \\
\hline
1 + 0 & & R_1 = 0 &
\end{array}
$$

Hence the required equation is

$$(13) \qquad x^3 - 7x + 6 = 0.$$

The student should verify this result by showing that the roots of (12) are 3, 4, -1 and that the roots of (13) are 1, 2, -3.

EXERCISES. GROUP 42

In each of Exs. 1–7, find the indicated root of the given equation correct to 1 decimal place, using the method of linear interpolation.

1. $x^3 - 3x^2 + 3x - 5 = 0$; $2 < x < 3$.

2. $x^3 - 6x^2 + 12x - 10 = 0;\ 3 < x < 4.$
3. $x^3 + 3x^2 + 2x - 7 = 0;\ 1 < x < 2.$
4. $x^3 + 3x^2 - 2x - 1 = 0;\ 0 < x < 1.$
5. $x^3 - 3x^2 - 26x + 69 = 0;\ 2 < x < 3.$
6. $x^4 - 2x^3 + 21x - 23 = 0;\ 1 < x < 2.$
7. $x^4 - 6x^3 + 12x^2 - 7x - 12 = 0;\ 3 < x < 4.$

8. By linear interpolation, find the positive root of $x^4 - 2x^3 - 3x^2 - 2x - 4 = 0$ correct to two decimal places.

9. By linear interpolation, find the negative root of $x^4 - 2x^3 - 3x^2 - 2x - 4 = 0$ correct to two decimal places.
Hint: Change the signs of the roots and find the corresponding positive root.

10. By linear interpolation, find the positive root of $4x^4 - 4x^3 + 7x^2 - 8x - 2 = 0$ correct to three decimal places.

11. Verify the result of Example 1 (Sec. 11.11).

In each of Exs. 12–15, transform the given equation into another having roots m times those of the given equation.

12. $x^3 - 2x^2 - x + 2 = 0;\ m = 3.$ 13. $2x^3 + x^2 - 13x + 6 = 0;\ m = 2.$
14. $x^4 - x^2 + x - 1 = 0;\ m = 3.$ 15. $x^3 + 3x^2 - 3x - 14 = 0;\ m = -2.$

16. By actually finding the roots, verify the result of Ex. 12.

17. Establish the corollary to Theorem 10 (Sec. 11.11).

In each of Exs. 18–21, transform the given equation into another whose roots are opposite in sign to those of the given equation.

18. $x^3 - 4x^2 + 14x - 20 = 0.$ 19. $2x^4 + 6x^3 - 7x^2 + 12 = 0.$
20. $3x^3 + 2x^2 - 9x + 2 = 0.$ 21. $x^4 - 3x^3 + 2x^2 - x + 1 = 0.$

22. By actually finding the roots, verify the result of Ex. 18.

In each of Exs. 23–26, transform the given equation into another whose roots are equal to those of the given equation multiplied by the smallest number which will make the leading coefficient unity and all other coefficients integers.

23. $4x^3 - 20x^2 + 9x + 28 = 0.$ 24. $2x^4 - 3x^3 - 14x^2 + 2x + 4 = 0.$
25. $3x^3 - x^2 - 3x + 1 = 0.$ 26. $2x^4 - 9x^3 + 10x^2 + x - 2 = 0.$

27. By actually finding the roots, verify the result of Ex. 23.

28. Verify the result of Example 3 (Sec. 11.11).

In each of Exs. 29–33, transform the given equation into another whose roots are less by the indicated number.

29. $3x^3 - 4x^2 - 35x + 12 = 0;\ 1.$ 30. $2x^3 - 9x^2 + 12x - 4 = 0;\ 3.$
31. $x^4 - 2x^3 - x^2 + 6x - 7 = 0;\ 2.$
32. $2x^4 + 6x^3 + 7x^2 + 2x - 2 = 0;\ 0.3.$
33. $2x^3 + 3x^2 + x - 1 = 0;\ 0.01.$

34. By actually finding the roots, verify the result of Ex. 29.

35. Transform the equation of Ex. 29 into another whose roots are greater by 1.

11.12. HORNER'S METHOD

We now consider the determination of irrational roots by a process known as *Horner's method of approximation.* This method is applicable only to rational integral equations, but it has the advantage that the computations are much simpler than in linear interpolation (Sec. 11.10). The ease of computation is due to the fact that each digit in the root is determined individually.

The reasoning underlying Horner's method is very simple. Say that a given rational integral equation $f(x) = 0$ has an irrational root which, correct to 3 decimal places, is 2.124. In order to determine this root we first establish the fact that the given equation has a root between 2 and 3 (Sec. 11.5). Next we diminish the roots of $f(x) = 0$ by 2, obtaining the new equation $f_1(x_1) = 0$ with the root 0.124 (Sec. 11.11). We then show that $f_1(x_1) = 0$ has a root between 0.1 and 0.2 and diminish its roots by 0.1, obtaining a new equation $f_2(x_2) = 0$ with the root 0.024. Repeating the previous step, we show that $f_2(x_2) = 0$ has a root between 0.02 and 0.03 and diminish its roots by 0.02, obtaining a new equation $f_3(x_3) = 0$ with the root 0.004. Continuing this process, we can obtain the root accurately to any number of decimal places. The details of the method are illustrated and explained best by means of an example.

Example. Show that the equation

$$f(x) = x^3 + 5x^2 - x - 9 = 0 \tag{1}$$

has a root between 1 and 2, and find it correct to 3 decimal places by Horner's method.

SOLUTION. By synthetic division we find $f(1) = -4$ and $f(2) = 17$ so that equation (1) has a root between 1 and 2.

We next diminish the roots of equation (1) by 1.

$$\begin{array}{rrrr|l}
1 & +5 & -1 & -9 & \underline{1} \\
 & +1 & +6 & +5 & \\ \hline
1 & +6 & +5 & \multicolumn{1}{|r}{-4} & \\
 & +1 & +7 & & \\ \hline
1 & +7 & \multicolumn{1}{|r}{+12} & & \\
 & +1 & & & \\ \hline
1 & \multicolumn{1}{|r}{\underline{+8}} & & & \\
\end{array}$$

The transformed equation

$$f_1(x_1) = x_1^3 + 8x_1^2 + 12x_1 - 4 = 0 \tag{2}$$

has a root between 0 and 1 which we now locate between two successive tenths. Since this root of (2) is small, its cube and square are still smaller, so that for a first approximation we may drop the terms in x_1^3 and x_1^2, thus obtaining the *trial equation* $12x_1 - 4 = 0$ with the solution $x_1 = 0.3^+$. Because this is only an approximation, we must test it in equation (2). By synthetic division we find $f_1(0.3) = 0.347$ and $f_1(0.2) = -1.272$. Hence equation (2) has a root between 0.2 and 0.3.

We therefore diminish the roots of equation (2) by 0.2. In this operation it is convenient, as indicated, to leave sufficient space for the necessary number of decimal places.

$$
\begin{array}{llll|l}
1 & +\,8.0 & +\,12.00 & -\,4.000 & \underline{0.2} \\
 & +\,0.2 & +\;\;1.64 & +\,2.728 & \\
\hline
1 & +\,8.2 & +\,13.64 & \multicolumn{1}{|l}{-1.272} & \\
 & +\,0.2 & +\;\;1.68 & & \\
\hline
1 & +\,8.4 & \multicolumn{1}{|l}{+15.32} & & \\
 & +\,0.2 & & & \\
\hline
1 & \multicolumn{1}{|l}{\underline{+8.6}} & & &
\end{array}
$$

The transformed equation

$$(3) \qquad f_2(x_2) = x_2^3 + 8.6x_2^2 + 15.32x_2 - 1.272 = 0$$

has a root between 0 and 0.1 which we locate between two successive hundredths. From the last two terms of (3), we have the trial equation $15.32x_2 - 1.272 = 0$ with the solution $x_2 = 0.08^+$. By synthetic division we find $f_2(0.08) = 0.009152$ and $f_2(0.07) = -0.157117$. Hence equation (3) has a root between 0.07 and 0.08.

We therefore diminish the roots of equation (3) by 0.07.

$$
\begin{array}{llll|l}
1 & +\,8.60 & +\,15.3200 & -\,1.272000 & \underline{0.07} \\
 & +\,0.07 & +\;\;0.6069 & +\,1.114883 & \\
\hline
1 & +\,8.67 & +\,15.9269 & \multicolumn{1}{|l}{-0.157117} & \\
 & +\,0.07 & +\;\;0.6118 & & \\
\hline
1 & +\,8.74 & \multicolumn{1}{|l}{+16.5387} & & \\
 & +\,0.07 & & & \\
\hline
1 & \multicolumn{1}{|l}{\underline{+8.81}} & & &
\end{array}
$$

The transformed equation

$$(4) \qquad f_3(x_3) = x_3^3 + 8.81x_3^2 + 16.5387x_3 - 0.157117 = 0$$

has a root between 0 and 0.01 which we locate between two successive thousandths. From the last two terms of (4), we have the trial equation

$16.5387x_3 - 0.157117 = 0$, with the solution $x_3 = 0.009^+$. By synthetic division we find $f_3(0.009) = -0.007554361$ and $f_3(0.01) = 0.009152$. Hence equation (4) has a root between 0.009 and 0.01.

We therefore diminish the roots of equation (4) by 0.009. It is left as an exercise to the student to show that the transformed equation is

$$(5) \qquad f_4(x_4) = x_4^3 + 8.837x_4^2 + 16.697523x_4 - 0.007554361 = 0,$$

from which the trial equation is $16.697523x_4 - 0.007554361 = 0$, with the solution $x_4 = 0.0004^+$. At this stage, since the root of (5) is very small, the solution of the trial equation is quite accurate. Hence the required root is

$$x = 1 + 0.2 + 0.07 + 0.009 + 0.0004 = 1.2794$$

and, accurate to 3 decimal places, is 1.279.

NOTES. 1. For expository purposes, the solution above is spread out. In actual practice the student should write the solution in compact form, showing only the operations of dimininishing the roots and omitting the transformed equations whose coefficients are already available.

2. It is highly important to test each successive digit of the required root to make certain that the root of each transformed equation lies between two successive values.

3. As we proceed further with the approximations in Horner's method, the roots of the transformed equations become smaller and smaller, so that the trial equations become more and more accurate and may often be used to obtain additional decimal places.

4. To find a negative root of $f(x) = 0$ by Horner's method, we find the corresponding positive root of $f(-x) = 0$ and change its sign.

EXERCISES. GROUP 43

In each of Exs. 1–6, find the indicated root of the given equation correct to two decimal places, using Horner's method.

1. $x^3 - 6x^2 + 13x - 13 = 0;\ 3 < x < 4.$
2. $x^3 - 3x^2 + 13x - 24 = 0;\ 2 < x < 3.$
3. $x^3 + 10x^2 + 34x - 60 = 0;\ 1 < x < 2.$
4. $x^3 - 10x^2 + 35x + 50 = 0;\ -1 > x > -2.$
5. $x^3 + 3x^2 - 5x - 47 = 0;\ 3 < x < 4.$
6. $x^3 - 9x^2 + 24x - 19 = 0;\ 2 < x < 3.$

In each of Exs. 7–11, find the indicated root of the given equation correct to three decimal places, using Horner's method.

7. $x^3 + 3x^2 + x - 6 = 0;\ 1 < x < 2.$
8. $x^3 - 4x^2 - 5x + 20 = 0;\ -2 > x > -3.$
9. $x^3 + 4x^2 + 6x - 97 = 0;\ 3 < x < 4.$

10. $x^4 + 4x^3 + 7x^2 - 2x - 21 = 0$; $1 < x < 2$.

11. $x^4 - 6x^3 + 12x^2 + 11x - 41 = 0$; $2 < x < 3$.

12. By Horner's method, solve Ex. 7 of Group 42 (Sec. 11.11).

13. By Horner's method, solve Ex. 8 of Group 42 (Sec. 11.11).

14. By Horner's method, solve Ex. 9 of Group 42 (Sec. 11.11).

15. By Horner's method, solve Ex. 10 of Group 42 (Sec. 11.11).

16. Show that decimals may be avoided in Horner's method by multiplying the roots of each transformed equation by 10.

17. In determining the root of the first transformed equation in Horner's method, a more accurate result may be obtained by using the last three terms as a trial equation. Show this for equation (2) in Sec. 11.12 by finding the positive root of the quadratic equation $8x_1^2 + 12x_1 - 4 = 0$.

18. Verify the transformed equation (5) of the Example of Sec. 11.12.

19. By Horner's method, find the positive root of $x^4 - 2x^3 - 9x^2 - 4x - 22 = 0$, correct to 3 decimal places.

20. By Horner's method, find the negative root of $x^4 - 2x^3 - 9x^2 - 4x - 22 = 0$, correct to 3 decimal places.

21. By Horner's method, find the positive root of $4x^4 - 19x^2 - 23x - 19 = 0$, correct to 3 decimal places.

22. By Horner's method, find the negative root of $4x^4 - 19x^2 - 23x - 19 = 0$, correct to 3 decimal places.

23. By Horner's method, find the principal cube root of 7, correct to 3 decimal places. *Hint:* Find the positive root of $x^3 - 7 = 0$.

In each of Exs. 24–27, find the principal indicated root, correct to 3 decimal places, by Horner's method.

24. $\sqrt[3]{15}$. 25. $\sqrt[3]{-35}$. 26. $\sqrt[4]{11}$. 27. $\sqrt[5]{27}$.

28. The dimensions of a rectangular box are 5 ft, 8 ft, and 9 ft. If each dimension is increased by the same amount, the volume is increased by 440 cu ft. By Horner's method, find the increase in each dimension.

29. Equal squares are cut from the corners of a rectangular piece of card board 14 in. long and 10 in. wide, and the remaining rectangular portions along the sides are folded up to form an open box whose volume is 100 cu in. By Horner's method, find the side of each square cut out. (Two solutions.)

30. By Horner's method, find the solutions of the system $x^2 + y = 7$, $y^2 + x = 11$, correct to 2 decimal places. Illustrate the results graphically.

11.13. RELATIONS BETWEEN ROOTS AND COEFFICIENTS

We have previously seen that the nature and value of the roots of a rational integral equation depend upon the coefficients. We will now obtain certain relations between the roots and coefficients of such an equation, relations which are often useful in its solution.

We will first form several equations with given roots. Thus, by Sec. 11.6, the equation whose roots are r_1 and r_2 is

$$(x - r_1)(x - r_2) = 0$$

or

$$x^2 - (r_1 + r_2)x + r_1r_2 = 0.$$

Similarly, the equation whose roots are r_1, r_2, and r_3 is

$$(x - r_1)(x - r_2)(x - r_3) = 0$$

or

$$x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3 = 0.$$

Similarly, the equation whose roots are r_1, r_2, r_3, and r_4 may be written in the form

$$x^4 - (r_1 + r_2 + r_3 + r_4)x^3 + (r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4)x^2 - (r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4)x + r_1r_2r_3r_4 = 0.$$

An examination of these equations discloses the following facts:

1. The leading coefficient is unity.
2. The coefficient of the second term is equal to minus the sum of all the roots.
3. The coefficient of the third term is equal to plus the sum of the products of the roots taken two at a time.
4. The coefficient of the fourth term is equal to minus the sum of the products of the roots taken three at a time.
5. The last term is equal to the product of all of the roots, the sign being plus or minus according as the number of roots is even or odd.

From these facts we may infer similar results for the general rational integral equation of degree n. That this inference is correct may be established by mathematical induction; we record it as

Theorem 12. *If $r_1, r_2, \cdots, r_n$ are the n roots of the rational integral equation*

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$$

with leading coefficient unity, then the roots and coefficients are connected by the following relations:

$$a_1 = -(r_1 + r_2 + \cdots + r_n),$$
$$a_2 = r_1r_2 + r_1r_3 + \cdots + r_{n-1}r_n,$$
$$a_3 = -(r_1r_2r_3 + r_1r_2r_4 + \cdots + r_{n-2}r_{n-1}r_n),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_n = (-1)^n r_1r_2r_3 \cdots r_n.$$

NOTES. 1. It is important to observe that the relations in Theorem 12 hold only when the leading coefficient is unity.

2. It will now be seen that Theorem 3 (Sec. 5.5) is the special case of Theorem 12 for $n = 2$.

Example 1. Solve the equation $3x^3 - 2x^2 - 27x + 18 = 0$ if one root is the negative of another root.

SOLUTION. Let the three roots be represented by r_1, $-r_1$, and r_2; their sum is equal to r_2.

Before applying Theorem 12, we divide the given equation through by 3 so as to obtain a leading coefficient of unity. Then the equation assumes the form

$$x^3 - \tfrac{2}{3}x^2 - 9x + 6 = 0,$$

and the sum of the roots is equal to $\frac{2}{3}$. Hence $r_2 = \frac{2}{3}$.

We may now depress the given equation by removing the root $\frac{2}{3}$, using synthetic division. Thus

$$\begin{array}{rrrr|l} 3 & -\,2 & -\,27 & +\,18 & \underline{\tfrac{2}{3}} \\ & +\,2 & +\,0 & -\,18 & \\ \hline 3 & +\,0 & -\,27 & +\,0 & \end{array}$$

The depressed equation is $3x^2 - 27 = 0$, with the solutions $x = \pm 3$. Hence the required roots are $\frac{2}{3}$, 3, -3.

Example 2. The roots of the equation $x^3 - 3x^2 + kx + 8 = 0$, in some order, are in arithmetic progression. Find these roots and the value of the coefficient k.

SOLUTION. We may represent the three roots by $a - d$, a, $a + d$; their sum is equal to $3a$. From the given equation, the sum of the roots is equal to 3. Hence $3a = 3$ and $a = 1$ is one of the roots.

By letting $x = 1$ in the given equation, we may obtain the value of k and then proceed to find the remaining two roots as in Example 1. However, we may obtain these roots without first finding k. Since $a = 1$, the three roots are $1 - d$, 1, $1 + d$, with the product $1 - d^2$. From the given equation, the product of the roots is equal to -8. Hence $1 - d^2 = -8$ and $d = \pm 3$. For $d = 3$, the roots are -2, 1, 4; for $d = -3$, they are 4, 1, -2.

The student may easily verify that $k = -6$.

EXERCISES. GROUP 44

1. Solve the equation $4x^3 - 12x^2 + 3x + 5 = 0$ if the roots, in some order, are in arithmetic progression.

2. Solve the equation $x^3 + 3x^2 - 6x - 8 = 0$ if the roots, in some order, are in geometric progression. *Hint:* Denote the roots by a/r, a, ar.

3. Solve the equation $x^3 - 9x^2 + kx - 24 = 0$ and find the value of k if the roots, in some order, are in arithmetic progression.

4. Solve the equation $3x^3 + kx^2 - 7x + 3 = 0$ and find the value of k if the roots, in some order, are in geometric progression.

5. Solve the equation $4x^3 - x^2 - 16x + 4 = 0$ if one root is the negative of another.

6. Solve the equation $x^3 - 10x^2 + 11x + 70 = 0$ if the sum of two of the roots is 3.

7. Solve the equation $x^3 + 2x^2 - 15x - 36 = 0$ if it has a double root.

8. Solve the equation $9x^3 - 45x^2 - 52x - 12 = 0$ if one root is double another.

9. Solve the equation $3x^3 + 17x^2 - 87x + 27 = 0$ if one root is the reciprocal of another.

10. Solve the equation $x^3 + 2x^2 - 5x - 6 = 0$ if one root exceeds another by 2.

11. Solve the equation $2x^3 + 9x^2 + 10x + 3 = 0$ if the roots are in the ratio 1:2:6.

12. Solve the equation $2x^3 - 11x^2 - 7x + 6 = 0$ if the product of two of its roots is 3.

13. Solve the equation $x^3 - 2x^2 - 5x + 6 = 0$ if the quotient of two of its roots is 3.

14. Solve the equation $x^4 - 5x^3 + 6x^2 + 4x - 8 = 0$ if it has a triple root.

15. Solve the equation $4x^4 + 28x^3 + 33x^2 - 56x + 16 = 0$ if it has two double roots.

16. Solve the equation $x^4 - 8x^3 + 14x^2 + 8x - 15 = 0$ if the roots, in some order, are in arithmetic progression. *Hint:* Denote the roots by $a - 3d$, $a - d$, $a + d$, $a + 3d$.

17. Solve the equation $9x^4 - 63x^3 + 53x^2 + 7x - 6 = 0$ if one root is the negative of another.

18. Write out the relations of Theorem 12 (Sec. 11.13) when the leading coefficient $a_0 \neq 1$.

19. For a rational integral equation, show that if the second term is lacking, the sum of the roots is zero, and that if the last (constant) term is lacking, at least one of the roots is equal to zero.

20. By considering the equation $x^n - 1 = 0$, show that (a) the sum of the n nth roots of unity is equal to zero; (b) the product of the n nth roots of unity is equal to -1 if n is even and is equal to $+1$ if n is odd. (See Ex. 17 of Group 29, Sec. 8.9.)

21. If r_1, r_2, and r_3 are the roots of the equation $6x^3 - 11x^2 - 3x + 2 = 0$, evaluate $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$ without actually finding the roots.

22. In Ex. 21, evaluate $r_1^2 + r_2^2 + r_3^2$ without actually finding the roots.

23. Find the relation which must exist among the coefficients of the equation $x^3 + ax^2 + bx + c = 0$ if one of its roots is to be the negative of another root. Verify this result for Example 1 of Sec. 11.13.

24. Find the relation which must exist among the coefficients of the equation $x^3 + ax^2 + bx + c = 0$ if the roots, in some order, form a geometric progression. Verify this result for Ex. 2.

25. Establish Theorem 12 (Sec. 11.13) by mathematical induction.

12

Partial fractions

12.1. INTRODUCTION

In Sec. 2.11 we considered the problem of finding the sum of two or more simple algebraic fractions. This sum was found to be a single fraction whose denominator is the least common multiple of the denominators of the given fractions. Thus, we may readily verify the following addition:

$$(1) \qquad \frac{1}{x+1} + \frac{2}{x-1} + \frac{2x-1}{x^2+1} = \frac{5x^3+x+2}{(x^2-1)(x^2+1)}.$$

In this chapter we consider the inverse problem, namely, that of resolving a given fraction into the sum of simpler fractions which are called its *partial fractions*. Thus, in relation (1), the three fractions in the left member are the partial fractions of the single fraction in the right member. The resolution of a given fraction into its partial fractions is required in other branches of mathematics, for example, in certain problems of integration in the calculus.

We have previously noted (Sec. 2.11) that an improper fraction may be expressed as the sum of a polynomial and a proper fraction. In the material following, it is to be understood that only proper fractions in their lowest terms will be resolved into partial fractions. Furthermore, we shall consider only fractions in which the numerator and denominator are polynomials with real coefficients. Since the denominators of the partial fractions to be determined are factors of the denominator of the given fraction, it follows that such denominator must have linear or irreducible quadratic factors with real coefficients, in accordance with Corollary 2 of Theorem 6 (Sec. 11.7).

12.2. THEOREM ON PARTIAL FRACTIONS

The method of resolving a given proper fraction into its partial fractions is based upon the following theorem, whose proof is omitted, for it is beyond the scope of this book.

Theorem. *Any given proper fraction, reduced to its lowest terms, may be expressed as the sum of partial fractions of the types classified as follows:*

1. *To each linear factor $ax + b$ which occurs only once as a factor of the denominator, there corresponds a partial fraction of the form $\frac{A}{ax + b}$, where $A \neq 0$ is a constant.*

2. *To each linear factor $ax + b$ which occurs k times as a factor of the denominator, there corresponds the sum of k partial fractions of the form*

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_k}{(ax + b)^k},$$

where $A_1, A_2, \cdots, A_k$ are constants and $A_k \neq 0$.

3. *To each quadratic factor $ax^2 + bx + c$ (irreducible in the field of real numbers) which occurs only once as a factor of the denominator, there corresponds a partial fraction of the form $\frac{Ax + B}{ax^2 + bx + c}$, where A and B are constants not both equal to zero.*

4. *To each quadratic factor $ax^2 + bx + c$ (irreducible in the field of real numbers) which occurs k times as a factor of the denominator, there corresponds the sum of k partial fractions of the form*

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k},$$

where $A_1, B_1, A_2, B_2, \cdots, A_k, B_k$ are constants and A_k and B_k are not both equal to zero.

NOTES. 1. If a given fraction is improper, it should first be expressed as the sum of a polynomial and a proper fraction. The theorem should then be applied to this proper fraction.

2. The types listed in the theorem are termed the *simplest partial fractions.*

3. The student may raise the question as to whether there are partial fractions of the form $\frac{Ax^2 + Bx + C}{ax^3 + bx^2 + cx + d}$. The answer is yes, but it is not the simplest partial fraction. Since we are dealing with real coefficients, it follows from Corollary 2 of Theorem 6 (Sec. 11.7) that the cubic denominator may be expressed either as the product of three linear factors or as the product of a linear factor and a quadratic factor. Hence the fraction above may be expressed as the sum of either two or three simpler fractions.

The theorem above gives the *form* of the partial fractions; it remains to determine the values of the various constants appearing in those fractions. In the remainder of this chapter we will explain the process by means of examples illustrating all four types.

12.3. LINEAR FACTORS, ALL DIFFERENT

Here we consider a problem involving type 1 of the theorem of Sec. 12.2.

Example. Resolve $\dfrac{5x+1}{(x-1)(x+1)(x+2)}$ into its simplest partial fractions.

SOLUTION. Since the factors of the denominator are all linear and different, it follows from the theorem that we may write the identity

$$(1) \qquad \frac{5x+1}{(x-1)(x+1)(x+2)} \equiv \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+2},$$

where A, B, and C are constants to be determined. Relation (1) holds for all values of x except 1, -1, and -2, for each of these values makes a denominator zero. Clearing (1) of fractions, we have the identity

$$(2) \quad 5x+1 \equiv A(x+1)(x+2) + B(x-1)(x+2) + C(x-1)(x+1),$$

which, in view of relation (1), holds for all values of x except possibly 1, -1, and -2. Hence, by the corollary of Theorem 5 (Sec. 11.6), relation (2) holds for *all* values of x including 1, -1, and -2.

There are two methods for determining the constants A, B, and C.

METHOD 1. To determine the three constants A, B, and C, we need three independent relations involving them. These three relations may be obtained by substituting any three distinct numerical values for x in the identity (2). However, it will be much simpler in this case if we substitute those values of x which were excluded from relation (1), namely, 1, -1, and -2, for each such substitution will eliminate all but one of the constants. Thus, for $x = 1$, the identity (2) becomes

$$5 + 1 = A(1+1)(1+2), \text{ whence } A = 1.$$

Similarly, for $x = -1$, the identity (2) becomes

$$-5 + 1 = B(-1-1)(-1+2), \text{ whence } B = 2.$$

Finally, for $x = -2$, the identity (2) becomes

$$-10 + 1 = C(-2 - 1)(-2 + 1), \text{ whence } C = -3.$$

Accordingly, we have as our required solution:

$$\frac{5x + 1}{(x - 1)(x + 1)(x + 2)} = \frac{1}{x - 1} + \frac{2}{x + 1} - \frac{3}{x + 2}. \tag{3}$$

A complete check of this result may be obtained by combining the three partial fractions in the right member of (3), as in Sec. 2.11.

METHOD 2. In this method, we expand the right member of (2) and collect the coefficients of the various powers of x. Thus,

$$5x + 1 \equiv A(x^2 + 3x + 2) + B(x^2 + x - 2) + C(x^2 - 1), \text{ or}$$

$$5x + 1 \equiv (A + B + C)x^2 + (3A + B)x + 2A - 2B - C. \tag{4}$$

Since (4) is an identity, it follows from Theorem 5 (Sec. 11.6) that the coefficients of like powers of x are equal, and we have

$$A + B + C = 0,$$
$$3A + B = 5,$$
$$2A - 2B - C = 1.$$

The solution of this system of equations (Sec. 4.7) is readily found to be $A = 1$, $B = 2$, $C = -3$, and is in agreement with the result of Method 1.

12.4. LINEAR FACTORS, SOME REPEATED

As an illustration of a problem involving type 2 of the theorem of Sec. 12.2, we have the following

Example. Resolve $\dfrac{5x^2 + 4x + 2}{(x - 4)(x + 3)^2}$ into its simplest partial fractions.

SOLUTION. This problem involves both types 1 and 2 of the theorem of Sec. 12.2, in accordance with which we write the identity

$$\frac{5x^2 + 4x + 2}{(x - 4)(x + 3)^2} \equiv \frac{A}{x - 4} + \frac{B}{x + 3} + \frac{C}{(x + 3)^2}. \tag{1}$$

Clearing (1) of fractions, we have the identity

$$5x^2 + 4x + 2 \equiv A(x + 3)^2 + B(x - 4)(x + 3) + C(x - 4), \tag{2}$$

which, by the same argument used in the example of Sec. 12.3, holds for all values of x.

Once again, there are two methods for determining the constants A, B, and C.

METHOD 1. Here, since one linear factor is repeated, it is not possible to obtain all three constants immediately by substituting certain values as in the example of Sec. 12.3. However, we may determine two of the constants in this manner. Thus for $x = 4$, the identity (2) becomes

$$80 + 16 + 2 = A(4 + 3)^2, \text{ whence } A = 2.$$

For $x = -3$, the identity (2) becomes

$$45 - 12 + 2 = C(-3 - 4), \text{ whence } C = -5.$$

There is no value that we can now substitute for x which will eliminate A and C and enable us to obtain B at once. However, if we use the values of A and C already obtained and some simple value of x, say 0, we can readily obtain B. Thus, if we substitute $A = 2$, $C = -5$, and $x = 0$ in the identity (2), we have

$$2 = 2(3)^2 + B(-4)(3) + (-5)(-4),$$

whence $$2 = 18 - 12B + 20, \quad 12B = 36, \quad B = 3.$$

Hence the required partial fractions are given by

$$\frac{5x^2 + 4x + 2}{(x - 4)(x + 3)^2} \equiv \frac{2}{x - 4} + \frac{3}{x + 3} - \frac{5}{(x + 3)^2}.$$

METHOD 2. We proceed here as in Method 2 of Sec. 12.3. Expanding the right member of (2), we have

$$5x^2 + 4x + 2 \equiv A(x^2 + 6x + 9) + B(x^2 - x - 12) + C(x - 4), \text{ or}$$

$$5x^2 + 4x + 2 \equiv (A + B)x^2 + (6A - B + C)x + 9A - 12B - 4C.$$

Equating the coefficients of like powers of x, we have the system

$$A + B = 5,$$

$$6A - B + C = 4,$$

$$9A - 12B - 4C = 2,$$

whose solution is readily found to be $A = 2$, $B = 3$, $C = -5$, and is in agreement with the result of Method 1.

EXERCISES. GROUP 45

In each of Exs. 1–20, resolve the given fraction into its simplest partial fractions and check the result.

1. $\dfrac{3x + 6}{(x - 2)(x + 4)}$.

2. $\dfrac{7x}{(2x + 1)(x - 3)}$.

3. $\dfrac{x - 9}{x^2 - 9}$.

4. $\dfrac{9x + 7}{x^2 + 2x - 3}$.

5. $\dfrac{3x^2 - 5x - 52}{(x + 2)(x - 3)(x + 5)}$.

6. $\dfrac{16 - 10x^2}{(x^2 - 1)(x^2 - 4)}$.

7. $\dfrac{-2x^2 + 14x + 18}{(x - 3)(2x^2 - x - 1)}$.

8. $\dfrac{2x^2 + x + 9}{x^3 - 2x^2 - 5x + 6}$.

9. $\dfrac{x^3 + 2x^2 - 1}{x^2 + x - 6}$.

10. $\dfrac{x^3 + 11x^2 + 37x + 31}{x^3 + 6x^2 + 5x - 12}$.

11. $\dfrac{3x - 1}{(x + 1)^2}$.

12. $\dfrac{x^2 + 3x - 2}{x^2(2x - 1)}$.

13. $\dfrac{9x^3 + 16x^2 + 3x - 10}{x^3(x + 5)}$.

14. $\dfrac{2x^3 + 7x^2 + 15x + 8}{x(x + 2)^3}$.

15. $\dfrac{3x^3 + 10x^2 - 5x}{(x - 1)^2(x + 1)^2}$.

16. $\dfrac{3x^3 + 4x^2 - 21x - 103}{(x - 3)(x^3 + 5x^2 - 8x - 48)}$.

17. $\dfrac{2x^3 + 3x^2 - 15x - 8}{(x + 2)(x^3 - 3x + 2)}$.

18. $\dfrac{4x^4 - 3x^2 + 6x - 3}{(x - 1)(x^2 - 1)^2}$.

19. $\dfrac{2x^4 - 4x^2 - x + 2}{(x^2 - x)^2}$.

20. $\dfrac{x^5 + 4x^4 - 15x^3 - 14x^2 + x + 24}{(x - 2)^2(x + 1)^3}$.

12.5. QUADRATIC FACTORS, ALL DIFFERENT

As an illustration of a problem involving type 3 of the theorem of Sec. 12.2, we have the following

Example. Resolve $\dfrac{3x^3 - x^2 + 4x}{(x^2 + 1)(x^2 - x + 1)}$ into its simplest partial fractions.

SOLUTION. Since both factors of the denominator of the given fraction are irreducible in the field of real numbers, we may write the following identity in accordance with the theorem of Sec. 12.2:

(1) $$\frac{3x^3 - x^2 + 4x}{(x^2 + 1)(x^2 - x + 1)} \equiv \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{x^2 - x + 1}.$$

Clearing (1) of fractions, we have the identity

(2) $3x^3 - x^2 + 4x \equiv (Ax + B)(x^2 - x + 1) + (Cx + D)(x^2 + 1).$

As before, there are two methods for determining the constants A, B, C, and D.

METHOD 1. In this method we substitute four different simple values for x in the identity (2). This gives us four independent relations involving the constants. Thus,

For $x = 0$, $0 = B + D$.

For $x = 1$, $6 = (A + B)(1) + (C + D)(2)$,

or $$A + B + 2C + 2D = 6.$$

For $x = -1$, $-8 = (-A + B)(3) + (-C + D)(2)$,

or $$3A - 3B + 2C - 2D = 8.$$

For $x = 2$, $24 - 4 + 8 = (2A + B)(3) + (2C + D)(5)$,

or $$6A + 3B + 10C + 5D = 28.$$

It is left as an exercise to the student to show that the solution of this system of four equations is $A = 1$, $B = -1$, $C = 2$, $D = 1$. Hence the required partial fractions are given by

$$\frac{3x^3 - x^2 + 4x}{(x^2 + 1)(x^2 - x + 1)} \equiv \frac{x - 1}{x^2 + 1} + \frac{2x + 1}{x^2 - x + 1}.$$

METHOD 2. The procedure here is the same as in Method 2 of the preceding article. Expanding the right member of (2), we have

$$3x^3 - x^2 + 4x \equiv Ax^3 - (A - B)x^2 + (A - B)x + B + Cx^3 + Dx^2 + Cx + D,$$

or $$3x^3 - x^2 + 4x \equiv (A + C)x^3 - (A - B - D)x^2 + (A - B + C)x + B + D.$$

Equating the coefficients of like powers of x, we have the system

$$\begin{aligned} A + C &= 3, \\ A - B - D &= 1, \\ A - B + C &= 4, \\ B + D &= 0, \end{aligned}$$

whose solution is found to be $A = 1$, $B = -1$, $C = 2$, $D = 1$, and is therefore in agreement with the result of Method 1.

12.6. QUADRATIC FACTORS, SOME REPEATED

As an illustration of a problem involving type 4 of the theorem of Sec. 12.2, we have the following

Example. Resolve $\dfrac{4x^4 + 13x^2 - 4x + 14}{(x-1)(x^2+2)^2}$ into its simplest partial fractions.

SOLUTION. In accordance with the theorem of Sec. 12.2, we may write the identity

$$(1) \qquad \frac{4x^4 + 13x^2 - 4x + 14}{(x-1)(x^2+2)^2} \equiv \frac{A}{x-1} + \frac{Bx + C}{x^2 + 2} + \frac{Dx + E}{(x^2+2)^2}.$$

Clearing (1) of fractions, we have the identity

$$(2) \qquad 4x^4 + 13x^2 - 4x + 14 \equiv A(x^2+2)^2 + (Bx + C)(x-1)(x^2+2) + (Dx + E)(x-1).$$

We have the same two methods of the preceding section for determining the constants A, B, C, D, and E.

METHOD 1. We have already noted (Sec. 12.3) that when a linear factor is involved, it is possible to substitute a particular value of x and determine one constant immediately. Thus, substituting $x = 1$ in the identity (2), we have

$$27 = 9A, \text{ whence } A = 3.$$

For the remaining constants, we will substitute simple values for x in the identity (2). Thus,

for $x = 0$, $A = 3$, $14 = 3(4) + C(-1)(2) + E(-1)$,

or

$$2C + E = -2.$$

For $x = -1$, $A = 3$,

$$35 = 3(3)^2 + (-B + C)(-2)(3) + (-D + E)(-2),$$

or

$$6B - 6C + 2D - 2E = 8.$$

For $x = 2$, $A = 3$,

$$64 + 52 - 8 + 14 = 3(6)^2 + (2B + C)(1)(6) + (2D + E)(1),$$

or

$$12B + 6C + 2D + E = 14.$$

For $x = -2$, $A = 3$,

$$64 + 52 + 8 + 14 = 3(6)^2 + (-2B + C)(-3)(6) + (-2D + E)(-3),$$

or

$$36B - 18C + 6D - 3E = 30.$$

The solution of this system of four equations is found to be $B = 1$, $C = 1$, $D = 0$, $E = -4$. Hence the required partial fractions are given by

$$\frac{4x^4 + 13x^2 - 4x + 14}{(x - 1)(x^2 + 2)^2} \equiv \frac{3}{x - 1} + \frac{x + 1}{x^2 + 2} - \frac{4}{(x^2 + 2)^2}.$$

METHOD 2. Expanding the right member of (2), we have

$$\begin{aligned} 4x^4 + 13x^2 - 4x + 14 \equiv{} & A(x^4 + 4x^2 + 4) + Bx^4 + (C - B)x^3 \\ & + (2B - C)x^2 + 2(C - B)x - 2C + Dx^2 \\ & + (E - D)x - E, \end{aligned}$$

or
$$\begin{aligned} 4x^4 + 13x^2 - 4x + 14 \equiv{} & (A + B)x^4 + (C - B)x^3 \\ & + (4A + 2B - C + D)x^2 \\ & + (2C - 2B - D + E)x + 4A - 2C - E. \end{aligned}$$

Equating the coefficients of like powers of x, we have the system

$$\begin{aligned} A + B &= 4, \\ C - B &= 0, \\ 4A + 2B - C + D &= 13, \\ 2C - 2B - D + E &= -4, \\ 4A - 2C - E &= 14, \end{aligned}$$

whose solution is found to be $A = 3$, $B = 1$, $C = 1$, $D = 0$, $E = -4$, and is therefore in agreement with the result of Method 1.

EXERCISES. GROUP 46

In each of Exs. 1–20, resolve the given fraction into its simplest partial fractions, and check the result.

1. $\dfrac{3x^2 - 4x + 5}{(x - 1)(x^2 + 1)}$.

2. $\dfrac{5x^2 + 8x + 5}{x^3 + 3x^2 + 3x + 2}$.

3. $\dfrac{2x^3 - 4x^2 + 4x - 4}{(x^2 + 1)(x^2 + 2)}$.

4. $\dfrac{3x^3 + x^2 + 2x - 2}{(x + 1)(x^3 + 1)}$.

5. $\dfrac{2x^2 + x + 3}{x^4 + 5x^2 + 6}$.

6. $\dfrac{-10x^2 - 24x - 48}{(x + 2)(x - 3)(x^2 + x + 2)}$.

7. $\dfrac{4x^3 + 3x^2 + 18x - 5}{(x + 1)(x^3 + 2x - 3)}$.

8. $\dfrac{3x^3 - 9x^2 + 8x - 10}{(x - 3)(x^3 - 2x^2 - x - 6)}$.

9. $\dfrac{2x^4 + 4x^3 + 4x^2 + x - 6}{x^4 + x^3 + 3x^2}$.

10. $\dfrac{x^5 + 7x^3 - x^2 + 9x - 12}{(x^2 + 3)(x^2 + x + 2)}$.

11. $\dfrac{x^3 + 2x^2 + 3x}{(x^2 + x + 1)^2}$.

12. $\dfrac{2x^5 + 4x^3 - 3x^2 + 3x - 1}{(x^2 + 1)^3}$.

13. $\dfrac{5x^5 - 13x^4 + 19x^3 - 22x^2 + 11x - 4}{(x^3 - x^2 + x)^2}$.

14. $\dfrac{7x^4 - 11x^3 + 12x^2 - 14x + 27}{(x - 3)^2(x^2 + 2)^2}$.

15. $\dfrac{2x^5 + 9x^3 + 3x^2 + 5x + 4}{x^6 + 2x^3 + 1}$.

16. $\dfrac{-4x^5 + 7x^4 - 4x^3 + 10x^2 + 7}{x^8 - 2x^4 + 1}$.

17. $\dfrac{5x^6 - 5x^5 + 6x^4 - 8x^3 + 5x^2 + 3x + 3}{(x - 1)(x^6 - 2x^3 + 1)}$.

18. $\dfrac{2x^7 - 7x^6 + 10x^5 - 16x^4 + 18x^3 - 16x^2 + 11x - 4}{(x^2 + 1)^2(x^2 - x + 1)^2}$.

19. $\dfrac{2x^9 + x^8 + 13x^7 + 10x^6 + 29x^5 + 24x^4 + 29x^3 + 18x^2 + 15x + 3}{(x^2 + 3)(x^2 + 1)^3}$.

20. $\dfrac{x^6 + 4x^5 + 11x^4 + 16x^3 + 21x^2 + 12x + 8}{(x^2 + 2)(x^2 + x + 2)^2}$.

13

Permutations and combinations

13.1. INTRODUCTION

In this chapter we shall study the various arrangements and selections it is possible to make from a given set of objects. While this will lead to the solution of some problems which are interesting in themselves, it will also include many practical applications. For example, we can find out how many different telephone numbers or automobile license plate numbers are possible, using a given set of letters and digits. Furthermore, in connection with combinations, we shall again consider binomial coefficients and Pascal's triangle (Secs. 7.5, 7.6). Finally, one significant aim of this chapter is to develop certain material required for the next chapter on the important subject of probability.

13.2. FUNDAMENTAL THEOREM

We first lay down the following

Definition. Each of the different arrangements which can be made by taking all or part of a number of things is called a *permutation*.

It should be noted that the *order* is paramount in a permutation. When we arrange the elements of a permutation in a certain order, we are said to *permute* those elements.

For example, let us exhibit the different arrangements or permutations which can be made from the three letters a, b, and c, taken two at a time. They are six in number, namely, ab, ac, ba, bc, ca, cb.

In the next section we shall derive a formula for the number of permutations of n different things taken r at a time. The proof is based on the following theorem, which, being basic, is appropriately called the *fundamental theorem.*

Theorem 1. (*Fundamental Theorem*). *If one act can be performed in any one of p different ways, and if, after this act is performed in any one of these ways, a second act can be performed in any one of q different ways, then the total number of different ways in which the two acts can be performed together in the order stated is pq.*

PROOF. With *each* of the p different ways in which the first act can be performed, the second act can be performed in q different ways, that is, there are q different ways of performing the two acts together for *each* way of performing the first act. Hence, for *all* of the p ways in which the first act can be performed, the two acts together can be performed in a total of pq different ways.

Corollary 1. *If one act can be performed in p different ways, a second act in q different ways, a third act in r different ways, and so on, then the total number of different ways in which all these acts can be performed together in the order stated is $pqr \cdots$*

Corollary 2. *If x acts can be performed successively in p different ways each, then the total number of different ways in which all of these x acts can be performed successively is p^x.*

As an illustration of Theorem 1, consider the example above listing the six permutations of the three letters a, b, c taken two at a time. We may regard this problem as one involving two successive acts consisting of filling two places or positions in order. The first place may be filled in three different ways by using each of the three letters a, b, c. After the first place is filled, two letters are left for the second place, which may therefore be filled in two different ways. Hence, by Theorem 1, both places may be filled in $3 \times 2 = 6$ different ways.

As we have previously noted, a formula for permutations will be derived in the next section. However, many problems may be solved without such a formula by using Theorem 1 and its corollaries and by considering the various acts to be performed as that of filling places or positions in order. We will illustrate the procedure by several examples.

Example 1. There are five roads between cities A and B, and four roads between cities B and C. Find the number of different ways a person may travel from A to C by way of B.

SOLUTION. We first write two short horizontal lines thus, —, —, to indicate the two places to be filled. The first place may be filled in five different ways since the first act of traveling from A to B may be performed in five different ways. Similarly, the second place may be filled in four different ways since the second act of traveling from B to C may be performed in four different ways. Our two places will now appear thus: $\underline{5}, \underline{4}$. Hence, by Theorem 1, the required number of different ways is $5 \times 4 = 20$.

Example 2. Find the number of different three-figure integers which can be formed from the digits 2, 3, 5, 7 if (a) repetitions are not allowed; (b) repetitions are allowed.

SOLUTION. (a) Consider, as in Example 1, that we have indicated the three places to be filled. The first place may be filled in four different ways. With the first place filled, the second place may be filled in three different ways by the remaining three digits. With the first two places filled, the third place may be filled in two different ways by the remaining two digits. Our three places will now appear thus: $\underline{4}, \underline{3}, \underline{2}$. Hence, by Theorem 1, the required number is the product $4 \times 3 \times 2 = 24$.

(b) If repetitions are allowed, the three filled places appear thus: $\underline{4}, \underline{4}, \underline{4}$, and the required number is the product $4 \times 4 \times 4 = 64$.

Example 3. In Example 2(a), how many of the integers are even?

SOLUTION. For even numbers, the third place (units) must be filled by the digit 2, and this can be done in only one way. From the remaining three digits, the first place (hundreds) may be filled in three different ways, and the second place (tens) may be filled in two ways. Hence, by Theorem 1, the total number of even integers is $3 \times 2 \times 1 = 6$.

EXERCISES. GROUP 47

1. Establish Corollary 1 of Theorem 1 (Sec. 13.2).

2. Establish Corollary 2 of Theorem 1 (Sec. 13.2).

3. Solve Example 3 (Sec. 13.2) if repetitions are allowed.

4. A building has 6 doors. In how many different ways can a person enter the building by one door and leave by a different door?

5. Find the number of different arrangements of the 4 letters a, b, c, d taken 3 at a time.

6. A club of 12 members is to elect a president, vice-president, secretary, and treasurer. How many different sets of officers may be chosen if each member is eligible for any office?

7. Solve Ex. 6 if 2 specific members only are eligible for the office of president but are also eligible for all the other offices.

8. Solve Ex. 6 if 2 specific members only are eligible for the office of president but are not eligible for any other office.

9. Find the number of different two-figure integers which can be formed from the digits 1, 2, 4, 7, 8 if (a) repetitions are not allowed; (b) repetitions are allowed.

10. In Ex. 9 find the number of even and the number of odd integers if (a) repetitions are allowed; (b) repetitions are not allowed.

11. Signals are formed from different colored flags by placing them one above the other on a pole. From 5 different flags, find the number of signals which can be formed by using (a) 3 of the flags; (b) 4 of the flags; (c) all of the flags.

12. In Ex. 11, find the total number of signals which can be formed by using one or more of the 5 flags.

13. A tossed coin can fall in either of 2 different ways, head or tail. Find the number of different ways in which the following number of coins can fall: (a) 2 coins; (b) 3 coins; (c) n coins.

14. The faces of a cubical die are numbered from 1 to 6 and hence, when thrown, can appear in any one of 6 different ways. Find the number of different ways in which the following number of dice can appear when thrown: (a) 2 dice; (b) 3 dice; (c) n dice.

15. If each of n dice has f faces numbered from 1 to f, find the number of different ways they can appear when thrown.

16. Find the number of four-letter words (not necessarily pronounceable) which can be formed from 10 different letters of the alphabet if (a) repetitions are not allowed; (b) repetitions are allowed.

17. Find the number of four-letter words which can be formed from 7 different consonants and 3 different vowels if the consonants and vowels are to alternate, repetitions not being allowed.

18. Solve Ex. 17 if repetitions are allowed.

19. In a certain state, automobile license plate designations consist of 5 places, the first 2 places being filled by any of the 26 letters of the alphabet and the last 3 places by any of the 10 digits from 0 to 9 inclusive, except that zero may not be used for the third place. Find the total number of different designations which can be formed if repetitions of letters and digits are not allowed.

20. Solve Ex. 19 if repetitions of both letters and digits are allowed.

21. Telephone number designations consist of 7 places, the first 2 places being filled by any of 24 specified letters of the alphabet and the last 5 places by any of the 10 digits from 0 to 9 inclusive, except that zero may not be used for either of the third and fourth places. Find the total number of different designations which can be formed if repetitions of letters and digits are not allowed.

22. Solve Ex. 21 if repetitions of digits only are allowed.

23. In how many different ways may 5 people be seated in a row of 8 chairs?

24. Solve Ex. 23 if the 5 people are to be seated in consecutive chairs.

25. Determine the total number of positive integers less than 5000 which can be formed from the 8 integers, 0 to 7 inclusive, repetitions not being allowed.

13.3. PERMUTATIONS

There are various symbols used to represent the number of permutations of n different things taken r at a time. We shall use the symbol $P(n, r)$, which is very appropriate because the number of permutations is a function of n and r. For the value of $P(n, r)$ we have

Theorem 2. *The number of permutations of n different things taken r at a time is given by the formula*

$$(1) \qquad P(n, r) = n(n-1)(n-2)\cdots(n-r+1), \quad r \leq n.$$

PROOF. The value of $P(n, r)$ is equal to the total number of ways in which r places can be filled by n different things. The first place can be filled in n different ways, since all n things are available. The second place can then be filled in $n - 1$ different ways with the remaining $n - 1$ things. Similarly, the third place can be filled in $n - 2$ different ways, and so on. Continuing this process, we see finally that the rth place can be filled in $n - (r - 1) = n - r + 1$ different ways. Then, by the fundamental theorem (Theorem 1, Sec. 13.2), the total number of ways is given by relation (1) above.

Corollary. *The total number of permutations of n different things taken all at a time is given by*

$$P(n, n) = n(n-1)(n-2)\cdots 1 = n! \text{ (Sec. 7.4)}$$

Example 1. How many different basket ball teams of 5 men each may be formed if 7 men are available to play any position?

SOLUTION. This problem may, of course, be solved by the fundamental theorem as in Sec. 13.2. However, we may also consider the result to be equal to the number of permutations of 7 things taken 5 at a time, which, by Theorem 2, is

$$P(7, 5) = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2520.$$

We next consider the case of determining the number of permutations of n things which are not all different. For example, let us determine P, the number of permutations of the five letters a, a, a, b, c, taken all at a time. Each of these P permutations contains the three identical letters a, a, a. If these three letters were all different from each other and from the remaining letters b, c, they could be permuted among themselves in $3!$ different ways for each of the P permutations, and all of the five different letters could be permuted in $5!$ ways. Hence $P \cdot 3! = 5!$, whence $P = \dfrac{5!}{3!} = 20$.

The general case is given by

Theorem 3. *If P represents the number of distinct permutations of n things taken all at a time, where p things are alike, q other things are alike, r other things are alike, and so on, then*

(2) $$P = \frac{n!}{p!\,q!\,r!\cdots}.$$

PROOF. If we first replace the p like things by p things different from each other and from the remainder of the n original things, then for each of the P permutations we can obtain $p!$ permutations by permuting the p new things among themselves. Hence, from the original P permutations, we thus obtain $P \cdot p!$ permutations each containing q like things, r like things, and so on. Similarly, by replacing the q like things by q different things, we obtain $P \cdot p!\,q!$ permutations, each containing r like things, and so on. Continuing this process, we finally obtain $P \cdot p!\,q!\,r!\cdots$ permutations, each consisting of n different things. But, by the corollary of Theorem 2, the number of such permutations is $n!$ Hence, $P \cdot p!\,q!\,r!\cdots = n!$, whence relation (2) follows.

Example 2. Find the number of different permutations which can be formed from the letters of the word *parallel*, taken all at a time.

SOLUTION. The word contains 8 letters, of which 3 are l, 2 are a, and the rest are all different. Hence, by Theorem 3, the number of different permutations is $\frac{8!}{3!\,2!} = \frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}{3\cdot 2\cdot 2} = 3360$.

We now consider the number of arrangements of n different things around a circle. Each such arrangement is called a *circular permutation.* If n different objects, one of which we shall designate as A, are arranged in a straight line, we have a different arrangement, depending on whether A is at the head or foot of the line, the remaining $n - 1$ objects having the same position in each case. This is not true, however, for a circular permutation, for the position of A may be considered fixed and the remaining $n - 1$ objects can then be arranged in $(n - 1)!$ different ways relative to A. We record this result as

Theorem 4. *The number of circular permutations of n different things is equal to $(n - 1)!$*

Example 3. A group of 3 girls and 3 boys is to be seated so that girls and boys alternate. Find the number of ways this may be done if (a) they are seated in a straight line; (b) they are seated around a table.

SOLUTION. (a) We may first consider the girls to be seated in the odd-numbered seats and the boys in the even numbered seats; this may be done in 3! 3! different ways. An equal number of different arrangements may be obtained by seating the boys in the odd-numbered seats and the girls in the even-numbered seats. Hence the total number of different ways is equal to $2 \cdot 3!\,3! = 72$.

(b) We may first seat the 3 girls around the table in 2! ways in accordance with Theorem 4. There are then three alternate places left in which to seat the 3 boys; this may be done in 3! ways. Hence the total number of different ways is equal to $2!\,3! = 12$.

EXERCISES. GROUP 48

1. Establish the corollary to Theorem 2 (Sec. 13.3).

2. Show that $P(n, r) = \dfrac{n!}{(n-r)!}$, $r \leq n$.

3. If repetitions are allowed, show that the number of permutations of n different things taken r at a time is equal to n^r.

4. Evaluate (a) $P(8, 2)$; (b) $P(9, 3)$.

5. Evaluate (a) $P(10, 4)$; (b) $P(7, 4) \div P(5, 4)$.

6. If $P(n, 4) = 6P(n, 2)$, find n.

7. If $P(n, 5) = 42P(n, 3)$, find n.

8. If $P(n, 5) = 24P(n, 2)$, find n.

9. If $2P(6, r) = 3P(5, r)$, find r.

10. If $12P(7, r) = 5P(9, r)$, find r.

11. Eight players are available for a basket ball team of 5 men. If 2 particular men can play center only and the remaining 6 men can play any position except center, find the number of different teams which can be formed.

12. Solve Ex. 11 if the 2 particular men can play any position.

13. Find the number of different baseball teams of 9 men each which can be formed from 15 available players of whom 3 can pitch only, 2 can catch only, 6 can play the infield only, and 4 can play the outfield only.

14. Find the number of different permutations which can be formed from the letters of the word *Alaska*, taken all at a time.

15. Signals are formed from 8 colored flags by placing them one above the other on a pole. Find the number of different signals which can be formed by using all 8 flags if 3 are red, 2 white, and the rest blue.

16. Solve Ex. 15 if the top flag must be red.

17. From a printing of one book, 6 copies are selected, and from a printing of another book, 5 copies are selected. Find the number of different ways these copies may be arranged on a shelf.

18. In how many different ways can 3 nickels, 4 dimes, and 5 quarters be distributed among 12 children, each to receive a coin?

19. There are m identical things of the first kind and n identical things of the second kind. Find the number of different permutations which may be formed, each containing p of the first kind and q of the second kind.

20. There are m copies of each of n different books. Find the number of ways in which they can be arranged on a shelf.

21. Find the total number of positive integers that can be formed from the digits 1, 2, 3, 4, 5, repetitions not being allowed, and show that the ratio of odd to even integers is 3:2.

22. Find the number of three-figure integers which can be formed from the 9 digits $1, 2, \cdots, 9$ if
(a) all three digits must be different.
(b) all three digits need not be different.
(c) all integers must be even, repetitions allowed.

23. In how many ways may 6 different books be arranged on a shelf if 2 particular books must be next to each other?

24. Solve Ex. 23 if 3 particular books must be next to each other.

25. Find the number of different ways 4 men and 3 women may be seated in a row of 7 chairs if the women are to be next to each other.

26. Solve Ex. 25 if 8 chairs are used.

27. In how many different ways may 5 different algebra texts and 4 different calculus texts be arranged on a shelf so that all books of one subject are together?

28. In how many different ways may 8 children stand in a circle?

29. In how many different ways may 8 differently colored beads be made into a necklace?

30. A group of 5 girls and 5 boys is to be seated in alternate chairs. Find the number of ways this may be done if (a) the chairs are in a straight line; (b) the chairs are around a circular table.

31. Six men, including A and B, are to speak at a meeting. In how many different orders may they speak?

32. Solve Ex. 31 if A must speak before B.

33. Seven people are to be seated in a row. Find the number of different ways this may be done if
(a) there are no restrictions.
(b) two particular people must sit next to each other.

34. Solve Ex. 33 if 2 particular people must not sit next to each other.

35. Solve Ex. 33 if the 7 people are to be seated in a circle.

13.4. COMBINATIONS

We first lay down the following

Definition. Each of the different groups which can be formed by taking all or part of a number of things, without reference to the order of the things in each group, is called a *combination*.

It should be noted that, in contradistinction to a permutation, the order of the constituents of a combination has no significance. Thus, while *ab* and *ba* are two distinct permutations, they represent one and only one combination, namely, the group consisting of the two letters *a* and *b*.

As in the case of the permutation, we have an appropriate symbol to represent the number of combinations of n different things taken r at a time. This symbol is $C(n, r)$; it is, of course, a function of n and r. Its value is given by

Theorem 5. *The number of combinations of n different things taken r at a time is given by the formula*

$$C(n, r) = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}, \qquad r \leq n. \tag{1}$$

PROOF. From each combination of r different things, we may form $r!$ permutations (Corollary, Theorem 2, Sec. 13.3). Hence, from all of the combinations, we may form a total of $C(n, r) \cdot r!$ permutations which is equal to $P(n, r)$, the number of permutations of n different things taken r at a time. Hence

$$C(n, r) \cdot r! = P(n, r),$$

whence

$$C(n, r) = \frac{P(n, r)}{r!},$$

which, from Theorem 2 (Sec. 13.3), may be written in the form

$$C(n, r) = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!},$$

and relation (1) is established.

It is worth noting in this last relation that numerator and denominator each has r factors.

Corollary 1. *The number of combinations of n different things taken all at a time is unity, that is,* $C(n, n) = 1$.

We will now obtain another form of relation (1) which is often more convenient. Thus, multiplying numerator and denominator of the right member of (1) by $(n - r)!$, we have

$$C(n, r) = \frac{n(n-1)\cdots(n-r+1)(n-r)!}{r!\,(n-r)!} = \frac{n!}{r!\,(n-r)!}.$$

Because of its importance, we state this result as

Corollary 2. *The number of combinations of n different things taken r at a time is also given by the formula*

$$C(n, r) = \frac{n!}{r!\,(n-r)!}, \qquad r \leq n. \tag{2}$$

Another important result may be obtained if we replace r by $n - r$ in relation (2). Thus,

$$C(n, n-r) = \frac{n!}{(n-r)!\,r!},$$

whence, from relation (2) we have

$$C(n, r) = C(n, n-r), \tag{3}$$

which we state as

Corollary 3. *The number of combinations of n different things taken r at a time is equal to the number of combinations of n different things taken n − r at a time.*

NOTE. The result of Corollary 3 could be anticipated because for each combination of r things selected from n different things, a corresponding group or combination of $n - r$ things is left. Such combinations are called *complementary*.

For example, if we select a committee of 3 from 9 persons, a corresponding (complementary) group of 6 persons is left.

Example 1. From 10 different consonants and 4 different vowels, how many words (not necessarily pronounceable) can be formed, each containing 6 consonants and 2 vowels?

SOLUTION. We first select 6 consonants from 10 consonants in $C(10, 6)$ ways. By relation (2), we have

$$C(10, 6) = \frac{10!}{6!\,4!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} = 210.$$

Similarly, we may select 2 vowels from 4 vowels in

$$C(4, 2) = \frac{4!}{2!\,2!} = \frac{4 \cdot 3}{1 \cdot 2} = 6 \text{ ways}.$$

Then for each of the 210 ways of selecting the consonants, we have 6 ways of selecting the vowels. Hence, by the fundamental theorem (Theorem 1, Sec. 13.2), the eight letters of each word can be selected in $210 \times 6 = 1260$ ways. After each such selection, the eight letters may be permuted in $8!$ different ways. Hence the total number of words that can be formed is $1260 \times 8! = 50{,}803{,}200$.

Example 2. From a group of 7 seniors and 6 juniors, a committee of 5 is to be chosen. Find the number of such committees containing (a) exactly 3 seniors; (b) at least 3 seniors.

SOLUTION. (a) In this case, there must be exactly 2 juniors. The seniors may be selected in $C(7, 3) = \dfrac{7!}{3!\,4!} = 35$ ways and the juniors in $C(6, 2) = \dfrac{6!}{2!\,4!} = 15$ ways. Hence, by the fundamental theorem, the total number of committees of 5 is $35 \cdot 15 = 525$.

(b) Here we have three types of committees: (1) three seniors and 2 juniors; (2) four seniors and 1 junior; (3) five seniors. The number of committees for each of the three types is then

(1) 525 committees by part (a).

(2) $C(7, 4) \cdot C(6, 1) = \dfrac{7!}{4!\,3!} \cdot 6 = 210.$

(3) $C(7, 5) = \dfrac{7!}{5!\,2!} = 21.$

Hence, adding, the total number of committees is $525 + 210 + 21 = 756$.

Example 3. Twelve points are coplanar but no three of them are collinear. (a) Find the number of different triangles which may be formed by using these points as vertices. (b) Find how many of these triangles have a particular point as a vertex.

SOLUTION. (a) Since each triangle has three vertices, we may form as many triangles as the number of ways in which we can select 3 points from 12 points. This number is $C(12, 3) = \dfrac{12!}{3!\,9!} = 220$.

(b) For a particular point to be a vertex of each triangle, we set that point aside and then select the other 2 points from the remaining 11 points in $C(11, 2) = \dfrac{11!}{2!\,9!} = 55$ different ways. Hence there are 55 triangles, each having a particular point as a vertex.

EXERCISES. GROUP 49

1. Establish Corollary 1 of Theorem 5 (Sec. 13.4).
2. Show that $C(n, n) = C(n, 0) = 1$.
3. Evaluate (a) $C(8, 4)$; (b) $C(7, 2)$.
4. Evaluate (a) $C(10, 8)$; (b) $C(18, 3)$.
5. If $C(n, 2) = 28$, find n.
6. If $C(n, 3) = 35$, find n.

7. If $2C(n, 5) = 3C(n, 3)$, find n. 8. If $C(n, 5) = 2C(n, 2)$, find n.

9. If $2C(6, r) = 3C(5, r)$, find r.

10. If $P(n, r) = 120$ and $C(n, r) = 20$, find n and r.

11. Show that $C(7, 3) = C(6, 3) + C(6, 2)$.

12. Show that $C(8, 5) - C(7, 5) = C(7, 4)$.

13. Find the number of committees of 4 which can be selected from a group of 15 persons.

14. In Ex. 13, find the number of committees which include a particular person.

15. In Ex. 13, find the number of committees which do not include a particular person.

16. Twelve points are coplanar but no 3 of them are collinear. Find the number of straight lines which can be drawn through these points.

17. In Ex. 16, find the number of lines which contain a particular point.

18. Find the number of diagonals of a convex polygon of 8 sides.

19. Find the number of "words," each containing 2 consonants and 2 vowels, which can be formed from 5 consonants and 3 vowels.

20. Solve Ex. 19 if consonants and vowels are to alternate.

21. There are 12 different books on a shelf. (a) Find the number of different selections of 8 books which may be made. (b) Determine the number of these selections which include a particular book. (c) Determine the number of these selections which include 2 particular books.

22. There are 15 points in space and no 4 of them are coplanar. (a) Find the number of planes determined by these points. (b) Determine the number of these planes containing one particular point. (c) Determine the number of these planes containing two particular points.

23. Find the number of committees, each consisting of 4 sophomores and 2 freshmen, which can be selected from 8 sophomores and 10 freshmen.

24. From a group of 6 men and 9 women, a committee of 5 is to be chosen. Find the number of such committees containing at least 2 women.

25. Solve Ex. 24 if the committees are to contain no more than 2 women.

26. A bag contains 3 white balls and 5 black balls. In how many ways may we select 3 balls so that: (a) exactly 2 are white; (b) at least 2 are white; (c) not more than 2 are white?

27. A bag contains 4 white, 2 black, and 3 red balls. In how many ways may we select 5 balls so that 2 are white, 1 is black, and 2 are red?

28. Solve Ex. 27 so that at least 3 balls are white in each selection of 5 balls.

29. In how many ways may a committee of 6 men be chosen from a group of 12 men if two particular men cannot serve on the same committee?

30. A company of 25 soldiers provides a guard of 3 men each night. Determine (a) the number of nights a different guard may be formed and (b) the number of nights each man will serve.

13.5. DIVISION INTO GROUPS

By Corollary 3 of Theorem 5 (Sec. 13.4), the number of combinations of n different things taken r at a time is equal to the number of combinations of n different things taken $n - r$ at a time. We then noted that for each combination containing r things, there is a complementary combination containing $n - r$ things. That is, the number of ways in which n different things may be divided into two groups, one containing r things and the other $n - r$ things, is given by relation (2) of Sec. 13.4:

$$C(n, r) = \frac{n!}{r!\,(n-r)!}\,. \tag{1}$$

We now extend this division to any number of groups. For convenience, let us start by considering the division of $p + q$ different things into two groups, one of p things and the other of q things, where $p \neq q$. By relation (1), the number of distinct ways, N_2, in which this can be done is

$$N_2 = \frac{(p+q)!}{p!\,q!}\,. \tag{2}$$

Next we consider the division of $p + q + r$ different things into three groups of p, q, and r things, respectively, where p, q, and r are unequal positive integers. First we divide $p + q + r$ things into two groups, one of p things and the other of $q + r$ things; by relation (1), this can be done in $\dfrac{(p+q+r)!}{p!\,(q+r)!}$ distinct ways. Similarly, each group of $q + r$ things can be divided into two groups, one of q things and the other of r things, in $\dfrac{(q+r)!}{q!\,r!}$ distinct ways. Then, by the fundamental theorem (Sec. 13.2), the total number of distinct ways of forming the three groups is

$$N_3 = \frac{(p+q+r)!}{p!\,(q+r)!}\cdot\frac{(q+r)!}{q!\,r!} = \frac{(p+q+r)!}{p!\,q!\,r!}\,. \tag{3}$$

In the same way, the results given by relations (2) and (3) may be extended to any number of groups. We state the general result as

Theorem 6. *Let $p, q, r, \cdots, t$ represent m unequal positive integers. Then the number of distinct ways of dividing $p + q + r + \cdots + t$ different things into m groups of $p, q, r, \cdots, t$ things, respectively, is*

$$N_m = \frac{(p+q+r+\cdots+t)!}{p!\,q!\,r!\cdots t!}\,.$$

Example 1. Find the number of ways 15 different books may be divided into three groups of 9, 4, and 2 books, respectively.

SOLUTION. By Theorem 6, the number of ways is

$$\frac{15!}{9!\,4!\,2!} = 75075.$$

Thus far we have considered only division into *unequal* groups. If the division is to be into *equal* groups, we must make a certain modification in Theorem 6. Say, for example, that we wish to divide 4 different things into two equal groups, each containing 2 things. If we use Theorem 6, the number of ways is $\frac{4!}{2!\,2!} = 6$ where, however, the two groups are permuted among themselves in 2! ways. We may exhibit this case by dividing 4 cards marked 1, 2, 3, 4 into two groups of 2 cards each. Thus, we have

Group 1	Group 2	
1, 2	3, 4	(1)
1, 3	2, 4	(2)
1, 4	2, 3	(3)
2, 3	1, 4	(3)
2, 4	1, 3	(2)
3, 4	1, 2	(1)

Note here that identical groups, but different in order, are designated by the same number at the right. Hence, if we consider the order of the groups, our result follows at once from Theorem 6; if however, the order of the groups is disregarded, we must divide the result of Theorem 6 by 2!, that is, the number of ways is $6/2! = 3$.

The preceding argument may be used for the general case of division into any number of equal groups. From Theorem 6, if we let $p = q = r = \cdots = t = n$, we obtain the number of ways of dividing mn different things into m groups of n things each, the order of the groups being regarded. If the order of these groups is disregarded, the last result must be divided by $m!$ We record these results as

Theorem 7. *The number of ways in which mn different things can be divided into m groups of n things each, where the order of the things in any group is not to be considered, is*

$\frac{(mn)!}{(n!)^m}$, *if the order of the groups is regarded;*

$\frac{(mn)!}{(n!)^m m!}$, *if the order of the groups is disregarded.*

The distinction noted in Theorem 7 is illustrated in

Example 2. A deck of playing cards consists of 52 different cards. Find (a) the number of ways four hands of 13 cards each may be dealt to four players in a game of bridge; (b) the number of ways the 52 cards may be divided into four groups of 13 cards each.

SOLUTION. (a) In a bridge game each different distribution of the hands among the players constitutes a different division. Hence, in this case, the groups are permuted, and by the first part of Theorem 7, the number of ways is $\frac{52!}{(13!)^4}$.

(b) In this case, the order of the groups is immaterial, and, by the second part of Theorem 7, the number of ways is $\frac{52!}{(13!)^4 4!}$.

13.6. NOTATION FOR SUMMATION

For the purposes of the next section it is convenient at this time to introduce a notation whereby it is possible to represent the sum of a sequence of terms in a very compact form. Thus, a sum of n terms such as $u_1 + u_2 + \cdots + u_n$ may be represented by the notation $\sum_{i=1}^{n} u_i$, where the symbol Σ is the capital Greek letter sigma and is called the *sign of summation*, while the letter i, called the *index of summation*, takes on successively all positive integral values from 1 to n inclusive. The symbol $\sum_{i=1}^{n} u_i$ is read "the summation of u_i from $i = 1$ to $i = n$."

Thus, in accordance with the notation for summation, we may write the sum of the terms of an arithmetic progression (Sec. 10.2) in the form

$$\sum_{k=1}^{n} [a_1 + (k-1)d] = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_1 + [n-1]d),$$

where each term in the right member is obtained by substituting 1, 2, 3, $\cdots$, n successively for k in the expression $a_1 + (k-1)d$. Note that the letter used for the index has no effect on the sum.

Similarly, a polynomial of degree n may be represented by this notation in the form

$$\sum_{k=0}^{n} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

13.7. BINOMIAL COEFFICIENTS

In Sec. 7.6, relation (2), it was shown that the $(r+1)$th term of the binomial expansion of $(a+b)^n$ is given by

(1) $$(r+1)\text{th term} = \frac{n(n-1)\cdots(n-r+1)}{r!}a^{n-r}b^r.$$

But by Theorem 5 (Sec. 13.4),

$$C(n, r) = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}, \qquad r \leq n,$$

so that (1) may be written in the form

(2) $$(r+1)\text{th term} = C(n, r)a^{n-r}b^r.$$

Hence, using the Σ notation of Sec. 13.6, we have

Theorem 8. *The entire binomial expansion may be written in the form*

(3) $$(a+b)^n = \sum_{r=0}^{n} C(n, r)a^{n-r}b^r.$$

The student may readily verify Theorem 8 by expanding the terms as given by the right member of (3), remembering that $C(n, 0) = C(n, n) = 1$ and that $0! = 1$ (Sec. 7.4). By evaluating the binomial coefficients, we obtain precisely the binomial theorem as given by relation (3) in Sec. 7.4.

In relation (3) above, let $a = b = 1$ and expand the right member. We obtain

$$(1+1)^n = C(n, 0) + C(n, 1) + C(n, 2) + \cdots + C(n, n).$$

Transposing $C(n, 0) = 1$, we have

(4) $$C(n, 1) + C(n, 2) + \cdots + C(n, n) = 2^n - 1.$$

We state relation (4) as

Theorem 9. *The total number of combinations of n different things taken* $1, 2, 3, \cdots, n$ *at a time is equal to* $2^n - 1$.

Example 1. Determine how many different sums of money can be made from a cent, a nickel, a dime, a quarter, a half dollar, and a dollar. (A single coin may be considered a "sum.")

SOLUTION. Here we have 6 different pieces of money. Hence, by taking them $1, 2, 3, \cdots, 6$ at a time, the total number of different sums of money we can form is $2^6 - 1 = 63$, in accordance with Theorem 9.

Once again, let $a = b = 1$ in relation (3) so that the right member will contain only the sum of the binomial coefficients, which we now write in the form:

$$(5) \qquad C(n, 0) + C(n, 1) + C(n, 2) + \cdots + C(n, n-2) + C(n, n-1) + C(n, n).$$

By Corollary 3 of Theorem 5 (Sec. 13.4), $C(n, r) = C(n, n - r)$. Hence, for the binomial coefficients in (5), we have $C(n, 0) = C(n, n)$, $C(n, 1) = C(n, n - 1)$, $C(n, 2) = C(n, n - 2), \cdots$, and so forth. In other words, we have here precisely the type of symmetry we noted as the fifth characteristic of the binomial expansion in Sec. 7.4. We record this result as

Theorem 10. *In the expansion of $(a + b)^n$, the binomial coefficients of any two terms equidistant from the ends of the expansion are equal.*

Because of the importance of the binomial coefficients, extensive tables of their values have been set up. In the construction of such tables, advantage is naturally taken of the symmetry noted in Theorem 10. In addition, use is made of the principle underlying the formation of Pascal's triangle, which was discussed in Sec. 7.5. This principle will now be proved as stated in

Theorem 11. (*Principle of Pascal's Triangle*). *In the binomial expansion of $(a + b)^n$, the coefficient of the $(r + 1)$th term is equal to the sum of the coefficients of the rth and $(r + 1)$th terms in the binomial expansion of $(a + b)^{n-1}$.*

PROOF. In order to establish this theorem, we must show, in accordance with relation (2), that

$$C(n, r) = C(n - 1, r - 1) + C(n - 1, r).$$

Thus, by Corollary 2 of Theorem 5 (Sec. 13.4), we have

$$\begin{aligned} C(n - 1, r - 1) &+ C(n - 1, r) \\ &= \frac{(n-1)!}{(r-1)!\,(n-r)!} + \frac{(n-1)!}{r!\,(n-r-1)!} \\ &= \frac{r(n-1)!}{r!\,(n-r)!} + \frac{(n-r)(n-1)!}{r!\,(n-r)!} = \frac{(r+n-r)(n-1)!}{r!\,(n-r)!} \\ &= \frac{n(n-1)!}{r!\,(n-r)!} = \frac{n!}{r!\,(n-r)!} = C(n, r). \end{aligned}$$

This completes the proof.

Example 2. By means of Theorem 11, find the binomial coefficients in the expansion of $(a + b)^6$ from the binomial coefficients in the expansion of $(a + b)^5$.

SOLUTION. Expanding $(a + b)^5$ by means of Theorem 8, we readily find its binomial coefficients to be, in order,

$$1, 5, 10, 10, 5, 1.$$

The first and last binomial coefficients in the expansion of $(a + b)^n$, n a positive integer, are each unity. Then by Theorem 11, the binomial coefficients of $(a + b)^6$, from the second on, are

$$\begin{aligned}
\text{2nd coefficient} &= 1 + 5 = 6,\\
\text{3rd coefficient} &= 5 + 10 = 15,\\
\text{4th coefficient} &= 10 + 10 = 20,\\
\text{5th coefficient} &= 10 + 5 = 15,\\
\text{6th coefficient} &= 5 + 1 = 6.
\end{aligned}$$

The last (7th coefficient) is, of course, unity. Also, due to the symmetry of the coefficients, it is only necessary to compute through the fourth coefficient.

In the various expansions of $(a + b)^n$, we observe that the binomial coefficients increase up to the middle of the expansion and then decrease in reverse order. From this we may infer that if n is even, with an odd number of terms in the expansion, the middle term has the greatest coefficient; and if n is odd, with an even number of terms in the expansion, the two middle terms each have the greatest coefficient. That this inference is correct follows from

Theorem 12. *The maximum value of $C(n, r)$ is obtained when $r = n/2$ if n is even, and when either $r = (n - 1)/2$ or $r = (n + 1)/2$ if n is odd.*

PROOF. By Theorem 8, $C(n, r)$ is the coefficient of the $(r + 1)$th term in the expansion of $(a + b)^n$. Hence, $C(n, r - 1)$ is the coefficient of the rth term, and we have the ratio:

$$\frac{C(n, r)}{C(n, r - 1)} = \frac{n!}{r!\,(n - r)!} \cdot \frac{(r - 1)!\,(n - r + 1)!}{n!} = \frac{n - r + 1}{r}.$$

Now the coefficient $C(n, r)$ is greater than its immediate predecessor $C(n, r - 1)$ as long as their ratio is greater than unity, that is, as long as

$$\frac{n - r + 1}{r} > 1,$$

whence $n - r + 1 > r$ and $r < \dfrac{n + 1}{2}$.

If n is even, $r = n/2$ is the greatest integer less than $(n + 1)/2 = n/2 + 1/2$.

If n is odd, $n - 1$ is even, and $r = (n - 1)/2$ is the greatest integer less than $(n - 1)/2 + 1 = (n + 1)/2$. But if n is odd, we may also have $r = (n + 1)/2$ since $C\left(n, \frac{n+1}{2}\right) = C\left(n, \frac{n-1}{2}\right)$ by Corollary 3 of Theorem 5 (Sec. 13.4).

This completes the proof.

EXERCISES. GROUP 50

1. Establish Theorem 6 (Sec. 13.5) by the method of proof used for Theorem 3 (Sec. 13.3).

2. Establish Theorem 7 (Sec. 13.5).

3. Consider the case of 6 cards, marked from 1 to 6, arranged into 2 hands of 3 cards each. Test Theorem 7 (Sec. 13.5) by exhibiting the actual arrangements of these hands.

4. Find the number of ways of dividing 9 different things into groups of 5 and 4 things. Compare this result with the number of ways of dividing 10 different things into 2 equal groups.

5. Show that the number of ways of dividing $2n - 1$ different things into groups of n and $n - 1$ things is equal to the number of ways of dividing $2n$ different things into 2 equal groups. Verify this result in Ex. 4.

6. Find the number of ways 12 different articles may be divided into 3 groups of 5, 4, and 3 articles, respectively.

7. Find the number of ways of dividing 12 different articles into 3 equal groups.

8. Find the number of ways of dividing 12 different articles equally among 3 people.

In each of Exs. 9–12, write out the terms of the sum indicated by the given summation.

9. $\sum_{i=1}^{n} u_i^3$. 10. $\sum_{i=1}^{n} a_i x_i$. 11. $\sum_{n=1}^{4} (2n - 1)$. 12. $\sum_{k=1}^{n} (-1)^{k+1} x^k$.

13. Show that the sum of a geometric progression of n terms whose first term is a_1 and whose common ratio is r may be represented by $\sum_{i=1}^{n} a_1 r^{i-1}$.

14. In how many different ways may a woman invite one or more of 5 friends to luncheon?

15. Find the number of different weights which may be weighed on a balance scale by using standard weights of $\frac{1}{4}$ lb, $\frac{1}{2}$ lb, 1 lb, and 2 lb on one arm.

16. From a group of 8 men, find the number of different committees which may be formed containing (a) one or more men; (b) two or more men.

17. Without actually expanding, find the sum of all the coefficients in the expansion of (a) $(a + b)^4$; (b) $(3a - b)^4$.

18. Without actually expanding, find the sum of all the coefficients in the expansion of $(x - 2y + 3z)^3$.

19. A coin is tossed 6 times. Find the number of different ways of obtaining (a) exactly 3 heads; (b) at least 3 heads; (c) at least 1 head.

20. Eight coins are tossed at one time. Find the number of different ways of obtaining (a) exactly 7 tails; (b) at least 7 heads; (c) at least 1 tail.

21. Without actually expanding, find the maximum coefficient in the expansion of $(a + b)^8$.

22. Without actually expanding, find the maximum coefficients in the expansion of $(a + b)^9$.

23. Show that the sum of the coefficients of the odd terms of a binomial expansion is equal to the sum of the coefficients of the even terms.

24. For the binomial expansion $(a + b)^n$, show that the coefficient of the middle term is even if n is even.

25. Show that $C(n, 1) + 2C(n, 2) + 3C(n, 3) + \cdots + nC(n, n) = n2^{n-1}$.

26. Verify Ex. 25 for $n = 4$.

27. Show that the general term in the expansion of $(a+b+c)^n$ is $\dfrac{n!}{p!q!r!}a^pb^qc^r$, where $p + q + r = n$.

28. Using the result of Ex. 27, find the coefficient of ab^2c^3 in the expansion of $(a + b + c)^6$.

29. Show that the number of ways in which the sum 7 can be obtained by throwing 2 six-faced dice is equal to the coefficient of x^7 in the expansion of $(x + x^2 + x^3 + x^4 + x^5 + x^6)^2$.

30. Using the method of Ex. 29, find the number of ways in which each of the sums from 2 to 12 inclusive can be obtained by throwing 2 six-faced dice. Check your results by showing that their sum is equal to 36.

14

Probability

14.1. INTRODUCTION

In this chapter we will give an elementary introduction to the theory of probability. The subject is so extensive in its applications and has assumed such great importance that entire treatises are devoted exclusively to it.

The mathematical theory of probability was founded about three centuries ago and was then concerned solely with games of chance. Since that time probability has been applied to a wide range of fields, several of which are mentioned here in order to give the student some idea of the importance of the subject.

One of the earliest applications of probability was in actuarial science, which considers life insurance and pension funds and their related problems. Another important use of probability is in statistics, which enters into a wide range of fields, such as finance, economics, biology, psychology, and the social sciences in general. Probability is also used in modern physics and chemistry. We note, finally, that probability has many uses in engineering, such as in the theory of least squares and the adjustment of observations, in problems of congestion (traffic problems), in the theory of sampling, and in the quality control of manufactured products.

Within the limits of this chapter we cannot, of course, discuss any of the above applications. But we shall consider some of the basic concepts of probability and a number of simple examples. Later, when the student has had further mathematical training, particularly in the calculus, he will be able to make a detailed study of one or more of the fascinating applications of probability.

14.2. DEFINITIONS

We are all familiar with the words "probability" and "chance" as used in everyday speech. Thus we say that it will probably snow tonight or that a certain plane will probably be late arriving at a designated air field. We note that both of these statements are predictions of the future and, as such, are short of certainty. They are also vague in the sense that they do not give any measure of the probability of the occurrence of either event. For our purposes in mathematics it will be necessary to frame a definition which will enable us to determine a numerical value or measure of the probability of the occurrence or nonoccurrence of a particular event. There are two definitions of probability in common use; we shall consider each of them in turn.

We know (Sec. 13.2) that if a cubical die is thrown, it can appear in any one of 6 different ways, each equally likely. The number 5 is one of these 6 different ways, and we then say there is 1 chance out of 6 that a 5 will appear. We also say that the probability of obtaining a 5 on one throw of a cubical die is $\frac{1}{6}$. Also, if we toss a coin, it can fall in either of 2 equally likely ways, a head or a tail, and by a similar argument, the probability of a head from one toss of a coin is said to be $\frac{1}{2}$. Note the term "equally likely" in both of these examples. It means, in the case of the die, that any one number is as likely to appear as another; in the case of the coin, the head is just as likely to appear as the tail. It is on the basis of such reasoning that we frame our first definition, which we record as

Definition 1. If an event can happen in a ways and fail in b ways, then the total number of all possible ways in which it can happen and fail is $a + b$. If each of these $a + b$ ways is *equally likely to occur*, then the probability p that the event will happen is *defined* as the ratio

$$p = \frac{a}{a + b}, \tag{1}$$

and the probability q that the event will fail is *defined* as the ratio

$$q = \frac{b}{a + b}. \tag{2}$$

In other words, the probability of the success of an event is defined as the ratio of the number of favorable cases to the total number of cases involved, each of these cases being considered equally likely to occur. Analogously, the probability of the failure of an event is defined as the ratio of the number of unfavorable cases to the total number of cases involved, each of these cases being considered equally likely to occur.

Thus, in the example above for the probability of a 5 appearing on one throw of a die, we have $a = 1$, $b = 5$, so that from (1) and (2) it follows that the probability of success is $p = 1/(1 + 5) = 1/6$ and the probability of failure is $q = 5/(1 + 5) = 5/6$.

Definition 1 is sometimes called the *classical definition* of probability. Since we know *beforehand* or *assume* that we know beforehand the number of favorable and unfavorable cases, Definition 1 is also said to be the *definition of a priori probability*.

From Definition 1 we may obtain a quantitative measure of probability. Of all the possible ways in which an event can happen and fail in one trial, it must either happen or fail once. This is defined as *certainty* and is readily found to be unity. Thus, by adding p and q as given by (1) and (2), we have

$$p + q = \frac{a}{a + b} + \frac{b}{a + b} = 1. \tag{3}$$

From (3), we obtain the following facts:

If the probability that an event will happen is p, then the probability that it will fail is $1 - p$. Thus, from (3),

$$q = 1 - p. \tag{4}$$

If the probability that an event will fail is q, then the probability that it will happen is $1 - q$. Thus, from (3),

$$p = 1 - q. \tag{5}$$

If an event is *certain to happen*, the probability that it will happen is 1, and the probability that it will fail is zero. Thus in (4), when $p = 1, q = 0$.

If an event is *certain to fail*, the probability that it will fail is 1, and the probability that it will happen is zero. Thus in (5), when $q = 1$, $p = 0$.

It is evident, therefore, that all values of probability lie between 0 and 1, and we write

$$0 \leq p \leq 1, \quad 0 \leq q \leq 1, \quad p + q = 1.$$

We now give several more definitions. As before, let a and b represent, respectively, the number of favorable and unfavorable cases for the occurrence of a particular event. If $a > b$, we say that *the odds are a to b in favor of the event;* if $a < b$, we say that *the odds are b to a against the event;* and if $a = b$, we say that *the chances are even.*

Thus, on one throw of a die there are four ways of obtaining 3 or more; hence the odds in favor of obtaining 3 or more are 4 to 2 or 2 to 1. Similarly, the odds are 5 to 1 against obtaining 6 on one throw of a die.

Let us now return to Definition 1. If the values of a and b are unknown, the definition cannot be used. This occurs in certain cases. For example, say that a company is about to manufacture an order of 2000 articles and desires to know how many may turn out defective. If we have no information about any previous experience with this operation, we cannot predict the number of defectives with any degree of confidence. However, say that by keeping accurate records in the previous manufacture of 100,000 of these same articles under the *same essential conditions*, it was observed that 1000 articles turned out defective. We then say that in any future operation of manufacturing these same articles under the *same essential conditions*, the probability of obtaining a defective in any one trial is equal to $\frac{1000}{100{,}000} = 0.01$. That is, we may expect one out of every 100 articles manufactured to be defective. Hence, for an order of 2000 articles we may expect $2000 \times 0.01 = 20$ defectives. The reasoning in this example is the motivation for

Definition 2. Consider an event which can either happen or fail in one trial. If it is observed that this event happens m times out of a total of n trials under the *same essential conditions*, then the ratio m/n is *defined* as the probability p that the event happens in any one trial, and we write

$$p = \frac{m}{n}. \tag{6}$$

We note that the term "same essential conditions" occurs in this definition and also in the preceding example. It means that each trial occurs (as nearly as possible) under precisely the same conditions. Thus, in the example of manufacturing certain articles, it means that the operation is performed by the same people using the same machines and equipment, all other conditions also being the same. It is, of course, questionable whether this can be realized in actual practice.

In Definition 2, called the *frequency definition*, the probability is really an estimate, and confidence in this estimate increases as n, the number of trials or observations, increases. For this reason, if the ratio m/n approaches a limit as n increases indefinitely (Sec. 10.5), this limit is also *defined* as the probability of the event happening in any one trial. We say that this is the result obtained *in the long run* or *on the average*.

Since probability, as given by Definition 2, is obtained from a large number of experiments or observations, it is often called *empirical* or *statistical probability*. Furthermore, in contradistinction to Definition 1, it is also called *a posteriori probability*.

From relation (6) we have $m = np$. Hence, if p is the probability of the occurrence of an event in a single trial, we say that the *frequency* or *expected number of occurrences* in n trials is equal to np. Thus, if the probability of a head in one toss of a coin is $\frac{1}{2}$, then in 100 tosses we may expect $100 \cdot \frac{1}{2} = 50$ heads. If np is not an integer, we take the integer closest to it in value.

The value of a given chance of winning a sum of money is called the *expectation*. If p is the probability that a person will win a sum of money s, his *expectation* is *defined* as ps. Thus in a lottery of 10 tickets for a single prize of \$50, the probability that one ticket will win the prize is $\frac{1}{10}$, and the expectation for one ticket is therefore $\frac{1}{10} \cdot \$50$ or \$5.

14.3. SIMPLE EVENTS

In this section we will consider some of the simplest types of problems in probability. Such types are associated with what are appropriately called *simple events* and for which we have the following

Definition. A *simple event* is a single event whose occurrence or non-occurrence is not involved with any other event.

Thus, a simple event is that of obtaining an ace with one throw of a six-faced die.

NOTE. Many problems in probability are concerned with coins, dice, and cards. Although the student is probably familiar with these terms, we will now describe them briefly in order that it may be clearly understood how they are used in the problems.

A coin has two distinct faces, one designated as a head and the other as a tail. The toss of a coin must always result in either a head or a tail appearing uppermost, each being equally likely.

A die (plural dice) is a small cube on each of whose six faces one or more dots appear, the number of dots designating the integers from 1 to 6 inclusive. When thrown, a die must always come to rest with one and only one face on top, each of the six faces being equally likely to appear. The number one is also called an *ace*, and the number two a *deuce*.

An ordinary deck of playing cards consists of 52 cards divided into 4 suits of 13 cards each. The names of the suits and their colors in parentheses are as follows: clubs (black), diamonds (red), hearts (red), and spades (black). Each suit consists of 9 cards numbered from 2 to 10 inclusive and 4 cards designated as ace, king, queen, and jack (in the order of descending value). The statement that a card is drawn "at random" means that the card is taken from a thoroughly shuffled deck so that any card is as likely to be chosen as another.

We now exhibit several typical examples.

Example 1. A coin is tossed 10 times. Find the probability that exactly 7 heads appear.

SOLUTION. Since a coin can appear in 2 different ways in one toss, in 10 tosses it can appear in 2^{10} ways (Cor. 2, Theorem 1, Sec. 13.2). From 10 heads, 7 heads may be selected in $C(10, 7) = 120$ different ways (Theorem 5, Sec. 13.4). Hence, by Definition 1 (Sec. 14.2), the required probability is

$$p = \frac{120}{2^{10}} = \frac{15}{128}.$$

Example 2. Find the probability of obtaining a sum of at least 10 on one throw of 2 dice, and determine the odds in favor of or against this event.

SOLUTION. One die can appear in 6 different ways; hence 2 dice can appear in $6 \cdot 6 = 36$ different ways. The sum 10 can be obtained in 3 ways: $5 + 5, 6 + 4, 4 + 6$; the sum 11 in 2 ways: $6 + 5, 5 + 6$; and the sum 12 in 1 way. Hence the total number of favorable ways is $3 + 2 + 1 = 6$, and by Definition 1, the required probability is $p = \frac{6}{36} = \frac{1}{6}$.

In this case $a = 6$ and $a + b = 36$; hence $b = 30$. Since $a < b$, the odds are 30 to 6 or 5 to 1 against this event.

Example 3. If 3 cards are drawn at random from a deck of 52 cards, find the probability that they are ace, king, and queen.

SOLUTION. From 52 cards, 3 may be selected in $C(52, 3)$ different ways (Sec. 13.4). Since there are 4 suits and each suit has an ace, king, and queen, these 3 cards may be obtained in $4 \cdot 4 \cdot 4$ different ways (Sec. 13.2). Hence, by Definition 1, the required probability is

$$p = \frac{4 \cdot 4 \cdot 4}{C(52, 3)} = \frac{4 \cdot 4 \cdot 4 \cdot 2 \cdot 3}{52 \cdot 51 \cdot 50} = \frac{16}{5525}.$$

Example 4. From a bag containing 4 white, 2 black, and 3 red balls, 5 balls are drawn at random. Find the probability that 2 are white, 1 is black, and 2 are red.

SOLUTION. From a total of $4 + 2 + 3 = 9$ balls, 5 balls may be selected in $C(9, 5)$ different ways (Sec. 13.4). Now 2 white balls may be selected from 4 white balls in $C(4, 2)$ ways, 1 black ball from 2 black balls in $C(2, 1)$ ways, and 2 red balls from 3 red balls in $C(3, 2)$ ways. Hence the total number of favorable ways is $C(4, 2) \cdot C(2, 1) \cdot C(3, 2)$ and, by Definition 1, the required probability is

$$p = \frac{C(4, 2) \cdot C(2, 1) \cdot C(3, 2)}{C(9, 5)} = \frac{6 \cdot 2 \cdot 3}{126} = \frac{2}{7}.$$

We next consider an example using Definition 2 of probability (Sec. 14.2). In computing their rates, insurance companies use a comprehensive table of observations known as a mortality table. Such a table gives a complete mortality record of a large number of persons, starting at an early age. For each year thereafter, the table shows the number of these persons still living. Thus, a table shows that of 1,000,000 persons living at 1 year of age, 941,806 persons are living at age 24 and that 906,554 persons are living at age 35. We then say, in accordance with Definition 2, that the probability of a person 1 year old living to be 35 is $\frac{906{,}554}{1{,}000{,}000}$ or about 0.91, and that the probability of a person aged 24 living to be 35 is $\frac{906{,}554}{941{,}806}$ or about 0.96.

Example 5. A mortality table shows that of 949,171 persons living at age 21, there are 577,882 living at age 65. (a) Find the probability that a man now 21 years of age will live to retire at 65. (b) Of a group of 2000 men now aged 21, find how many may be expected to retire at 65.

SOLUTION. (a) By Definition 2, the probability of attaining age 65 is $p = \frac{577{,}882}{949{,}171} = 0.609$ (about).

(b) For $n = 2000$ and $p = 0.609$, the expected number of occurrences (Sec. 14.2) $= np = 2000(0.609) = 1218$, and this is the number of men who may be expected to retire at 65.

EXERCISES. GROUP 51

In the following exercises, p and q represent, respectively, the probability of the success and the probability of the failure of an event.

1. If the odds are in favor of an event happening, show that $p > \frac{1}{2}$. If the odds are against an event happening, show that $p < \frac{1}{2}$. If the chances of an event happening are even, show that $p = q = \frac{1}{2}$.

2. Show that the ratio of the odds in favor of an event happening is equal to p/q and that the ratio of the odds against an event happening is equal to q/p.

3. The probability that an event will happen is $\frac{3}{5}$. Find the odds in favor of the event.

4. The probability that an event will fail is $\frac{6}{11}$. Find the odds against the event.

5. The odds are 7 to 3 in favor of an event. Find the probability that the event will fail.

6. The odds are 5 to 4 against an event. Find the probability that the event will happen.

7. Find the probability of obtaining a sum of 7 on one throw of 2 dice, and determine the odds against this event.

8. Find the probability of obtaining a sum of 7 or less on one throw of 2 dice, and determine the odds in favor of this event.

9. A coin is tossed 4 times. Find the probability that exactly 3 heads appear.

10. For one toss of 4 coins find the probability that (a) exactly 3 heads appear; (b) at least 3 heads appear.

11. One card is drawn at random from a deck of 52 cards. Find the probability that (a) it is a club; (b) it is either a diamond or a heart; (c) it is not a spade.

12. If 4 cards are drawn at random from a deck of 52 cards, find the probability that they are ace, king, queen, and jack.

13. From a bag containing 7 white and 5 black balls, 6 balls are drawn at random. Find the probability that 4 are white and 2 are black.

14. In Ex. 13, find the probability that of the 6 balls drawn, at least 4 are white.

15. From a group of 8 boys and 6 girls, a committee of 4 is to be chosen by lot. Find the probability that it will consist of 2 boys and 2 girls.

16. In Ex. 15, find the probability that the committee will have no more than 2 boys.

17. A mortality table shows that of 557,882 persons living at age 65, the number living at age 80 is 181,765. Of 1000 men retiring at age 65, find how many are expected to be alive 15 years later.

18. A person is to receive a prize of $90 if he obtains a sum of 9 or more on one throw of 2 dice. Find the value of his expectation.

19. A person is to receive a prize of $51 if he obtains a spade and a diamond when drawing 2 cards at random from a deck of 52 cards. Find the value of his expectation.

20. A man with 2 dimes and 2 nickels in his pocket draws out 2 coins at random to pay a 15-cent bus fare. Find the probability that he draws out the exact fare.

21. Nine different books are arranged at random on a shelf. Find the probability that 3 particular books are next to each other.

22. Nine people are seated at random in a circle. Find the probability that 2 particular people are next to each other.

23. From a bag containing 6 white, 4 black, and 2 red balls, 6 balls are drawn at random. Find the probability that 3 are white, 2 are black, and 1 is red.

24. From a bag containing 5 white balls, 3 black balls, and 1 red ball, 3 balls are drawn at random. Find the probability that no white ball is drawn.

25. Find the probability of obtaining a sum of 15 on one throw of 3 dice.

26. Find the probability of obtaining a sum of at least 15 on one throw of 3 dice.

27. On one throw of 3 dice, find the probability that the same number appears on any 2 but only 2 of the dice.

28. Two cards are drawn at random from a pack of 10 cards numbered 1 to 10. Find the probability that the sum of the numbers on the drawn cards is (a) even; (b) odd.

29. A man draws one card at random from a pack of 10 cards numbered from 1 to 10, where the number on each card represents the same number of dollars as a prize. Find the value of his expectation.

30. On one throw of a die, a man is to receive an amount in dollars equal to the number thrown. Find the value of his expectation.

31. A man draws 3 coins at random from a bag containing 8 dimes, 4 quarters, and 3 half-dollars. Find the value of his expectation.

32. A coin has been tossed 6 times and 6 successive heads have appeared. Find the probability of head on the next toss.

33. If 3 cards are drawn at random from a deck of 52 cards, find the probability that they are (a) all hearts; (b) all of the same suit.

34. If 3 cards are drawn at random from a deck of 52 cards, find the probability that 1 is a heart and 2 are diamonds.

35. If 5 women and 4 men are seated at random in a row, find the probability that the women and men sit in alternate seats.

36. If 4 women and their husbands are seated at random in a row of 8 chairs, find the probability that each woman is seated next to her husband.

37. If 4 cards are drawn at random from a deck of 52 cards, find the probability that they include one from each suit.

38. If 5 cards are drawn at random from a deck of 52 cards, find the probability that they include exactly 3 kings.

39. If 5 cards are drawn at random from a deck of 52 cards, find the probability that they include exactly 3 of the same denomination.

40. A *yarborough* is a bridge hand of 13 cards, none of which is higher than a 9. Show that the odds against holding a yarborough are 1827 to 1.

14.4. COMPOUND EVENTS

Here we will consider problems which, in general, are somewhat more difficult than those of the preceding section. This is due to the fact that we will now study the probability of *compound events*, that is, the occurrence of two or more simple events. Compound events may be classified into three types: independent events, dependent events, and mutually exclusive events. We will define and discuss each of these types in turn.

Definition. Two or more events are said to be *independent* if the occurrence of any one of them *does not affect* the probability of the occurrence of any of the other events.

Thus, the compound event of obtaining *both* an ace on one throw of a die *and* a head from a single toss of a coin is composed of two independent events, for the occurrence of an ace on the die does not affect the probability of the appearance of a head on the coin and vice-versa.

We will now establish an important fundamental theorem.

Theorem 1. (*Multiplication Theorem for Independent Events*). *If p_1 and p_2 are the respective probabilities of the occurrence of two independent* *events E_1 and E_2, then $P = p_1p_2$ is the probability that both E_1 and E_2 occur simultaneously or in turn.*

PROOF. As in Definition 1 (Sec. 14.2), let $p_1 = a_1/(a_1 + b_1)$, where a_1 is the number of ways E_1 can happen and b_1 is the number of ways it can fail, all of these ways being equally likely. Similarly, let $p_2 = a_2/(a_2 + b_2)$, where a_2 is the number of ways E_2 can happen and b_2 is the number of ways it can fail. Then by Theorem 1 (Sec. 13.2), the total number of ways in which both E_1 and E_2 can happen is a_1a_2, and the total number of ways in which they can both happen and fail is $(a_1 + b_1)(a_2 + b_2)$. Then by Definition 1 (Sec. 14.2), the probability P that *both E_1 and E_2* can happen simultaneously or in turn is

$$P = \frac{a_1a_2}{(a_1 + b_1)(a_2 + b_2)} = \frac{a_1}{a_1 + b_2} \cdot \frac{a_2}{a_2 + b_2} = p_1p_2.$$

This completes the proof.

Corollary 1. *If $p_1, p_2, \cdots, p_n$ are the respective probabilities of the occurrence of n independent events, then $P = p_1p_2 \cdots p_n$ is the probability that all of these events occur simultaneously or in turn.*

Corollary 2. *If p is the probability that an event will happen in any single trial, then the probability that it will happen r times in succession, or in r specified trials, is p^r.*

NOTES. 1. It should be observed that in both the statement and proof of Theorem 1 (appropriately called the *Multiplication Theorem*), the words *both—and* are used. Thus, we speak of the occurrence of *both* events E_1 *and* E_2. These words are characteristic of problems in independent events.

2. When we speak of independent events occurring in turn, we mean that they may occur in *any order*.

Example 1. Find the probability of obtaining *both* a deuce on one throw of a die *and* a tail from a single toss of a coin.

SOLUTION. The probability of a deuce is $\frac{1}{6}$; the probability of a tail is $\frac{1}{2}$. Since these are independent events, it follows from Theorem 1 that the probability of obtaining *both* a deuce *and* a tail is $\frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$.

Example 2. The probabilities that A and B can solve a particular problem are $\frac{2}{3}$ and $\frac{3}{4}$, respectively. Find the probability that the problem will be solved if both try it.

SOLUTION. There are several ways of working out this example. We will use a short and simple method.

The problem will be solved if *both A and B* do *not* fail to solve it.

The probability that A will fail is $1 - \frac{2}{3} = \frac{1}{3}$; the probability that B will fail is $1 - \frac{3}{4} = \frac{1}{4}$. Hence, by Theorem 1, the probability that *both A and B* will fail is $\frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$. Hence, the probability that both A and B do *not* fail is $1 - \frac{1}{12} = \frac{11}{12}$, and this is the probability that the problem will be solved.

Definition. Two or more events are said to be *dependent* if the occurrence of any one of them affects the probability of the occurrence of any of the other events.

Thus, consider the probability of obtaining 2 clubs in two successive drawings from a deck of 52 cards. The probability of a club in the first drawing is $\frac{13}{52} = \frac{1}{4}$. If the first card drawn is replaced, the probability of a club in the second drawing is again $\frac{13}{52} = \frac{1}{4}$. These two events are independent and hence, in accordance with Theorem 1, the probability of obtaining 2 clubs is $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$. If, however, the first card drawn is a club and is *not* replaced, there are 12 clubs remaining out of 51 cards, so that the probability of obtaining a club in a second drawing is $\frac{12}{51} = \frac{4}{17}$. In this case the probability of the second drawing is dependent on the first drawing. By the same reasoning used in establishing Theorem 1 on independent events, we may show that the probability of the occurrence of these two dependent events is equal to the probability of the first drawing multiplied by the probability of the second drawing. Hence, the probability of obtaining 2 clubs in this case is $\frac{1}{4} \cdot \frac{4}{17} = \frac{1}{17}$.

We now state the theorem on dependent events. Since the proof is similar to that of Theorem 1, we leave the details to the student as an exercise.

Theorem 2. (*Multiplication Theorem for Dependent Events*). *Let p_1 be the probability of an event E_1 whose occurrence affects the probability p_2 of the occurrence of a second event E_2. Then $P = p_1p_2$ is the probability that both E_1 and E_2 occur in that order.*

NOTE 3. In Theorem 2, p_2 must represent the probability that E_2 happens *after* E_1 has happened.

Example 3. If 2 balls are drawn in succession from a bag containing 4 white and 3 black balls, find the probability p that the first ball drawn is white and the second black if (a) the first ball drawn is replaced; (b) the first ball drawn is not replaced.

SOLUTION. (a) The probability of a white ball in the first drawing is $\frac{4}{7}$. If this ball is replaced, the probability of a black ball in the second drawing is $\frac{3}{7}$. Hence, by Theorem 1, $p = \frac{4}{7} \cdot \frac{3}{7} = \frac{12}{49}$.

(b) As before, the probability of a white ball in the first drawing is $\frac{4}{7}$. If this ball is not replaced, the probability of a black ball in the second drawing is $\frac{3}{6} = \frac{1}{2}$. Hence, by Theorem 2, $p = \frac{4}{7} \cdot \frac{1}{2} = \frac{2}{7}$.

Definition. Two or more events are said to be *mutually exclusive* when the occurrence of any one of them excludes the occurrence of any other.

Thus, the compound event of obtaining *either* an ace *or* a 3 on one throw of a die is composed of two mutually exclusive events, because if the ace appears, the 3 cannot appear, and vice versa. We can easily compute the probability of this event by the methods of Sec. 14.3. Thus an ace and a 3 may each appear in one way, and we thus have two favorable cases. Since there are a total of 6 cases, the required probability is $\frac{2}{6} = \frac{1}{3}$ by Definition 1 (Sec. 14.2). The student will note that this result is the sum of the individual probabilities ($\frac{1}{6}$) of obtaining an ace and a 3, that is, $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$. This is an illustration of

Theorem 3. (*Addition Theorem for Mutually Exclusive Events*). *The probability P that either one or another of any number of mutually exclusive events should occur is the sum of the probabilities of the occurrence of the separate events.*

PROOF. Let there be r mutually exclusive events whose separate probabilities of occurrence are $p_1, p_2, \cdots, p_r$, respectively. Then we are to show that

$$P = p_1 + p_2 + \cdots + p_r.$$

Suppose that out of a total of n ways in which an event can either happen or fail, the first event can happen in a ways, the second event in b ways, $\cdots$, and the rth event in k ways, all of these ways being equally likely to occur. Since all these events are mutually exclusive, all these ways of occurrence are different, and we therefore have a total of $a + b + \cdots + k$ ways in which *either* one event *or* another can occur. Hence by Definition 1 (Sec. 14.2), we have

(1) $$p_1 = \frac{a}{n}, \; p_2 = \frac{b}{n}, \cdots, p_r = \frac{k}{n},$$

and

(2) $$P = \frac{a + b + \cdots + k}{n}.$$

Now (2) may be written

$$P = \frac{a}{n} + \frac{b}{n} + \cdots + \frac{k}{n}$$

From (1) $$= p_1 + p_2 + \cdots + p_r.$$

This completes the proof.

NOTE 4. It should be observed that in both the statement and proof of Theorem 3, the words *either—or* are used. These words are characteristic of problems in mutually exclusive events.

Example 4. One bag contains 4 white and 2 black balls; a second bag contains 2 white and 5 black balls. If a single ball is drawn from one of the bags selected at random, find the probability that it is white.

SOLUTION. The probability of selecting the first of the two bags is $\frac{1}{2}$, and the probability of drawing a white ball from it is $\frac{4}{6}$ or $\frac{2}{3}$. Hence, by Theorem 2, the probability of *both* selecting the first bag *and* then drawing a white ball from it is $\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$. Similarly, the probability of *both* selecting the second bag *and* then drawing a white ball from it is $\frac{1}{2} \cdot \frac{2}{7} = \frac{1}{7}$. Since the white ball must be drawn from *either* the first *or* second bag, it follows from Theorem 3 that the required probability is $\frac{1}{3} + \frac{1}{7} = \frac{10}{21}$.

Many problems in probability may be solved by more than one method. In the following group of exercises the student will find it instructive to solve a problem by one method and then, if possible, check the result by using another. Some problems may be solved by using the methods of either Sec. 14.3 or Sec. 14.4. For example, under the definition of dependent events, we found that the probability P of obtaining 2 clubs in two successive drawings from a deck of 52 cards is equal to $\frac{1}{17}$ if the first card drawn is *not* replaced. This result may also be obtained as in Sec. 14.3, namely, that $P = \dfrac{C(13, 2)}{C(52, 2)} = \dfrac{1}{17}$.

EXERCISES. GROUP 52

1. If the probabilities that r independent events will happen are $p_1, p_2, \cdots, p_r$, respectively, prove that the probability that all of them will fail is

$$(1 - p_1)(1 - p_2) \cdots (1 - p_r).$$

2. Prove Theorem 1 (Sec. 14.4) by using the frequency definition (Definition 2 Sec. 14.2). *Hint:* Use the fact that if p is the probability of the occurrence of an event in a single trial, the expected number of occurrences in n trials is equal to np.

3. Establish Corollary 1 of Theorem 1 (Sec. 14.4).

4. Establish Corollary 2 of Theorem 1 (Sec. 14.4).

5. Establish Theorem 2 (Sec. 14.4).

6. State and prove a corollary to Theorem 2 which is analogous to Corollary 1 of Theorem 1 (Sec. 14.4).

7. Prove Theorem 3 (Sec. 14.4) by using the frequency definition (Definition 2, Sec. 14.2).

8. A single die is thrown twice. Find the probability of obtaining an ace on the second throw but not on the first.

9. A single die is thrown 3 times. Find the probability of obtaining an ace on the second throw only.

10. A coin is tossed 4 times. Find the probability of obtaining a head on the third toss only.

11. Find the probability of obtaining exactly 3 heads from 4 tosses of a coin and the sum of 11 on one throw of 2 dice.

12. The probabilities that A and B can solve a particular problem are $\frac{1}{2}$ and $\frac{3}{5}$, respectively. Find the probability that the problem will be solved if both try it.

13. The probabilities that A, B, and C can solve a particular problem are $\frac{1}{2}$, $\frac{3}{5}$, and $\frac{2}{3}$, respectively. Find the probability that the problem will be solved if all three try it.

14. A, B, and C are firing at a target and their respective probabilities of hitting it are $\frac{2}{7}$, $\frac{2}{5}$, and $\frac{2}{3}$. If all three fire at the target, find the probability that it will be hit.

15. Solve Example 2 (Sec. 14.4) by considering the following mutually exclusive events: both A and B succeed; A succeeds and B fails; B succeeds and A fails.

16. Solve Example 2 (Sec. 14.4) by the following chain of reasoning. If A tries first, the probability that the problem will be solved by A is $\frac{2}{3}$; call this Event 1. If A fails, the probability that B will try to solve the problem is $1 - \frac{2}{3} = \frac{1}{3}$, and hence the probability that the problem will be solved by B is $\frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$; call this Event 2. Then consider Events 1 and 2 as mutually exclusive.

17. Solve Example 2 (Sec. 14.4) by the method of Ex. 16, but let B try first.

18. Solve Ex. 12 by each of the methods of Exs. 15, 16, and 17.

19. Two balls are drawn in succession from a bag containing 2 white and 4 black balls. Find the probability that the first ball drawn is white and the second black if (a) the first ball drawn is replaced; (b) the first ball drawn is not replaced.

20. In Ex. 19, if 2 balls are drawn at random, find the probability that they are both of the same color.

21. In Ex. 19, if 2 balls are drawn at random, find the probability that one is white and the other is black. Add the probabilities obtained in both Exs. 20 and 21, and account for the sum.

22. One bag contains 2 white and 6 black balls; a second bag contains 5 white and 3 black balls. If a single ball is drawn from each bag, find the probability that both balls are of the same color.

23. In Ex. 22, find the probability that the 2 balls are of different color.

24. In Ex. 22, if a single ball is drawn from one of the bags selected at random, find the probability that it is white.

25. In Ex. 24, find the probability that the ball drawn is black.

26. In Ex. 22, a single ball is drawn from the first bag and placed in the second bag. The balls in the second bag are then thoroughly mixed and a single ball is drawn from this bag. Find the probability that this ball is white.

27. In Ex. 26, find the probability that the last ball drawn is black.

28. In Ex. 22, a single ball is drawn from the second bag and placed in the first bag. The balls in the first bag are then thoroughly mixed and a single ball is drawn from this bag. Find the probability that this ball is (a) white; (b) black.

29. If 3 cards are drawn at random from a deck of 52 cards, find the probability that they are all of the same color.

30. If 3 cards are drawn in succession from a deck of 52 cards, each card being replaced before the next one is drawn, find the probability that they are all of the same color.

31. A mortality table shows that the probabilities of A and B living 25 years longer are 0.9 and 0.8, respectively. Find the probability that at the end of 25 years (a) both are alive; (b) both are dead; (c) A is alive and B is dead; (d) A is dead and B is alive. Add these results and account for the sum.

32. The probabilities that A and B can solve a particular problem are $\frac{1}{2}$ and $\frac{2}{3}$, respectively. Find the probability that (a) both solve the problem; (b) both fail to solve the problem; (c) A solves the problem but B fails; (d) B solves the problem but A fails. Add the results and account for the sum.

33. A, B, and C are entered in 3 separate races, and the probabilities that each wins his own race are $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$, respectively. Find the probability that (a) no one wins his own race; (b) exactly 1 wins his race; (c) exactly 2 win their races; (d) all 3 win their races. Add the results and account for the sum.

34. One die is a regular tetrahedron, that is, a regular solid having 4 faces marked 1, 2, 3, 4; another die is the conventional cube having 6 faces marked from 1 to 6 inclusive. For one throw of these 2 dice, find the probability that the sum appearing is greater than 7.

35. The probabilities of a favorable review of a manuscript by 3 independent readers are $\frac{3}{5}$, $\frac{2}{3}$, and $\frac{3}{4}$, respectively. Find the probability that a majority of the 3 reviews will be favorable.

36. A and B each throw a single die once, the one first obtaining an ace to win a prize. If A throws first, find their respective probabilities of winning.

37. A and B, in that order, throw a single die alternately until one wins by throwing an ace. Find the probability each has of winning. *Hint:* Use infinite geometric series (Sec. 10.5).

38. A, B, and C, in order, cut a deck of 52 cards, that is, each one draws a card at random and then replaces it. If the first one who cuts a diamond wins, find their respective probabilities of winning.

39. A, B, and C, in order, toss a coin until one wins by tossing a head. If the prize for winning is \$35, find their respective expectations.

40. A and B, in that order, throw a pair of dice alternately until either A wins by throwing 6 or B wins by throwing 7. Find the probability each has of winning.

14.5. REPEATED TRIALS

In this section we consider the problem of repeated trials, a topic of fundamental importance in the theory of probability and its applications. This problem arises when an experiment or observation is repeated a number of times under the same conditions.

We have previously used the term *trial* without formally defining it (Sec. 14.2). Now, for clarity of exposition and precision of statement, we will define this term and several others.

A simple event is said to undergo a *single trial* if it must either happen or fail once.

Thus, one toss of a coin constitutes a single trial since the coin must fall either a head or a tail once. We note that a trial has essentially the characteristics of an experiment.

A simple event is said to undergo a *repeated trial* if it must either happen or fail once under *exactly the same conditions* as any other trial of the event.

Thus, two or more tosses of a coin constitute repeated trials, for in each toss the coin must fall either a head or a tail once under exactly the same conditions.

If an event happens in a single trial, it is convenient to say that the event *succeeds*, that the trial is a *success*, and that the probability of the event happening is the *probability of success*. Similarly, if an event fails to happen in a single trial, we say that the event *fails*, that the trial is a *failure*, and that the probability of the event failing to happen is the *probability of failure*.

As a simple introduction to our next theorem, let us consider

Example 1. Determine the probability P of obtaining exactly 3 aces in 5 throws of a single die.

SOLUTION. Each throw of a single die is a trial. Let the act of obtaining an ace be considered a success. Then, in a single trial, the probability of success is $\frac{1}{6}$ and the probability of failure is $1 - \frac{1}{6} = \frac{5}{6}$. We are then to determine the probability of exactly 3 successes in 5 trials.

The probability of success in a single trial being $\frac{1}{6}$, the probability of 3 successes is $(\frac{1}{6})^3$ in any 3 specified trials (Cor. 2, Theorem 1, Sec. 14.4). As there are 5 trials, 3 successes *must be accompanied by* 2 failures. The probability of one failure being $\frac{5}{6}$, the probability of 2 failures is $(\frac{5}{6})^2$. Hence, by the multiplication theorem (Theorem 1, Sec. 14.4), the probability of *both* 3 successes *and* 2 failures is $(\frac{1}{6})^3(\frac{5}{6})^2$. But these 3 successes may occur in *any* 3 of the 5 trials. Thus, 3 aces may appear *either* on the first 3 throws *or* on the second, fourth, and fifth throws, and so forth. That is, we may obtain 3 aces in as many different ways as we can select 3 things from 5 things, or in $C(5, 3)$ ways. Since these different ways are mutually exclusive, it follows by the addition theorem (Theorem 3, Sec. 14.4) that our required probability is the sum of $C(5, 3)$ terms each equal to $(\frac{1}{6})^3(\frac{5}{6})^2$, that is,

$$P = C(5, 3) \cdot \left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2 = \frac{5 \cdot 4}{1 \cdot 2} \cdot \frac{1}{6^3} \cdot \frac{25}{6^2} = \frac{125}{3888}.$$

The preceding example is an illustration of the general theorem which we state as

Theorem 4 (*Binomial Law*). *For a single trial of an event, let p be the probability of success and $q = 1 - p$ the probability of failure. Then the probability P_1 of exactly r successes in n repeated trials is given by the formula*

$$(1) \qquad P_1 = C(n, r)p^r q^{n-r}, \qquad r \leq n.$$

PROOF. The probability that the event will happen in r specified trials is p^r and that it will fail in the remaining $n - r$ trials is q^{n-r} (Cor. 2, Theorem 1, Sec. 14.4). The probability of *both* r specified successes *and* the accompanying $n - r$ failures is then $p^r q^{n-r}$ (Theorem 1, Sec. 14.4). But r successes may be selected from n trials in $C(n, r)$ different ways all of which are equally likely and mutually exclusive. Hence, by Theorem 3 (Sec. 14.4), the required probability P_1 is given by the formula (1) above.

NOTE 1. By relation (2) of Sec. 13.7, we see that P_1, as given by relation (1) above, is the $(n - r + 1)$th term in the binomial expansion of $(p + q)^n$. Theorem 4 is therefore often called the *Binomial Law*.

By means of Theorem 4 we may easily establish

Theorem 5. *For a single trial of an event, let p be the probability of success and $q = 1 - p$ the probability of failure. Then the probability P_2 of at least r successes in n repeated trials is given by the relation*

$$(2) \qquad P_2 = \sum_{r=n}^{r=r} C(n, r)p^n q^{n-r}, \qquad r \leq n.$$

PROOF. If the event is to happen *at least* r times in n trials, it must happen *either exactly* n times, *or exactly* $n - 1$ times, *or exactly* $n - 2$ times, $\cdots$, *or exactly* r times. In other words, we have the following $n - r + 1$

mutually exclusive events:

Event No.	Happens Exactly	Probability by Theorem 4
1	n times $= n - (1) + 1$	$C(n, n)p^n q^{n-n} = p^n$
2	$n - 1$ times $= n - (2) + 1$	$C(n, n-1)p^{n-1}q$
3	$n - 2$ times $= n - (3) + 1$	$C(n, n-2)p^{n-2}q^2$
. . .	. . .	. . .
$n - r + 1$	r times $= n - (n - r + 1) + 1$	$C(n, r)p^r q^{n-r}$

Adding these probabilities by the addition theorem (Theorem 3, Sec. 14.4), we have

$$(3) \qquad P_2 = p^n + C(n, n-1)p^{n-1}q + C(n, n-2)p^{n-2}q^2 + \cdots + C(n, r)p^r q^{n-r},$$

which, in view of our sigma notation (Sec. 13.6), may be written in the form of relation (2) above.

NOTE 2. By Theorem 8 of Sec. 13.7, the right member of (3) represents the first $n - r + 1$ terms in the binomial expansion of $(p + q)^n$.

Example 2. A coin is tossed 8 times. Find the probability that at least 6 heads appear.

SOLUTION. For a single toss, the probability of a head is $p = \frac{1}{2}$; hence the probability of a tail is $q = 1 - p = \frac{1}{2}$. In this problem the number of trials is $n = 8$. Then in accordance with Theorem 5, the required probability P_2 is the sum of the probabilities of obtaining exactly 8 heads, exactly 7 heads, and exactly 6 heads. Hence,

$$P_2 = C(8, 8)\left(\frac{1}{2}\right)^8 + C(8, 7)\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right) + C(8, 6)\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^2$$

$$= \frac{1}{2^8} + 8 \cdot \frac{1}{2^8} + \frac{8 \cdot 7}{1 \cdot 2} \cdot \frac{1}{2^8} = \frac{37}{256}.$$

Example 3. From a deck of 52 cards, a single card is drawn at random. The card is then replaced, the deck is thoroughly shuffled, and again a single card is drawn at random. This process is performed a total of six times. Find the probability of obtaining at least 1 heart.

SOLUTION. The student may at first be inclined to solve this problem by the method used in the preceding example, that is, to add up the six separate probabilities of obtaining exactly 1 heart, exactly 2 hearts, $\cdots$, exactly 6 hearts. But the same result may be obtained much more easily if we compute the probability of *failing* to get a heart six times in succession and then subtract this probability from unity.

The probability of obtaining a heart in one draw or trial is $\frac{13}{52} = \frac{1}{4}$; hence the probability of failing to get a heart in one trial is $1 - \frac{1}{4} = \frac{3}{4}$. Then the probability of failing to obtain a heart in six successive trials is $(\frac{3}{4})^6$. Hence the probability of *not* failing to get a heart in six successive trials is $1 - \left(\frac{3}{4}\right)^6 = 1 - \frac{729}{4096} = \frac{3367}{4096}$, and this is the probability of obtaining at least 1 heart in six trials.

NOTE 3. The student will see in the next section that the required probability in Example 3 is the sum of all the terms except one in a binomial expansion whose value is unity. Hence it is much easier to obtain the required result by computing the one exceptional term and then subtracting this value from unity.

14.6. BINOMIAL EXPANSION

It was observed in both Theorems 4 and 5 of the preceding section that the various probabilities appearing in the problem of repeated trials are terms in the binomial expansion of $(p + q)^n$, where p is the probability of success and $q = 1 - p$ is the probability of failure in each of n trials. By Theorem 8 (Sec. 13.7), this expansion may be expressed as

$$(q + p)^n = \sum_{r=0}^{n} C(n, r)q^{n-r}p^r,$$

which we may write out in the form

$$(1) \quad (q + p)^n = C(n, 0)p^0q^n + C(n, 1)pq^{n-1} + C(n, 2)p^2q^{n-2} + \cdots + C(n, n - 1)p^{n-1}q + C(n, n)p^nq^0,$$

where $C(n, 0) = 1$, $C(n, 1) = n, \cdots, C(n, n) = 1$ are the usual binomial coefficients. The terms of this expansion, taken in order, represent, respectively, for n trials, the probabilities of exactly no successes and n failures, 1 success and $n - 1$ failures, 2 successes and $n - 2$ failures, $\cdots$, n successes and no failures. Hence these terms represent the probabilities of all possible cases and since these events are mutually exclusive, their sum must be equal to unity. That this is so follows from the fact that since $q + p = 1$, $(q + p)^n = 1$.

In general, the successive terms in the expansion (1) increase up to a certain value (or possibly two equal values) and then decrease. This term is called the *maximum* and has the property that its ratio to the preceding term and to the following term is greater than or equal to unity in each case. We will now determine this maximum term. More specifically, we will determine the value of r (number of successes) for which the general term

$C(n, r)q^{n-r}p^r$ in the binomial expansion of $(q + p)^n$ is a maximum. We have first the ratios

(2) $$\frac{(r+1)\text{th term}}{r\text{th term}} \geq 1,$$

(3) $$\frac{(r+1)\text{th term}}{(r+2)\text{th term}} \geq 1.$$

From (2) we have

$$\frac{C(n, r)q^{n-r}p^r}{C(n, r-1)q^{n-r+1}p^{r-1}} = \frac{p}{q} \cdot \frac{n!}{r!\,(n-r)!} \cdot \frac{(r-1)!\,(n-r+1)!}{n!}$$

$$= \frac{p}{q} \cdot \frac{n-r+1}{r} \geq 1,$$

whence $$np - pr + p \geq qr.$$

Since $q = 1 - p$, $$np - pr + p \geq r - pr, \quad \text{or}$$

(4) $$np + p \geq r.$$

From (3) we have

$$\frac{C(n, r)q^{n-r}p^r}{C(n, r+1)q^{n-r-1}p^{r+1}} = \frac{q}{p} \cdot \frac{n!}{r!\,(n-r)!} \cdot \frac{(r+1)!\,(n-r-1)!}{n!}$$

$$= \frac{q}{p} \cdot \frac{r+1}{n-r} \geq 1,$$

whence $$qr + q \geq np - pr.$$

Since $qr = (1 - p)r$, $$r - pr + q \geq np - pr, \quad \text{or}$$

(5) $$r \geq np - q.$$

Hence, from (4) and (5), we have

(6) $$np + p \geq r \geq np - q.$$

From (6) we see that the integer r lies between two values which differ from each other by unity, for $p + q = 1$. We record this result as

Theorem 6. *For a single trial of an event, let p be the probability of success and $q = 1 - p$ the probability of failure. Then in n repeated trials, the number of successes r having the greatest probability of occurrence is an integer which lies between $np + p$ and $np - q$.*

It is usual to take np as the value of r giving the maximum probability. From Theorem 6 we have the following appropriate

Definition. The *most probable value* of the number of successes r in n repeated trials is an integer to which corresponds a greater probability of occurrence than to any other value of r. Its value is approximately equal to np where p is the probability of success in a single trial.

We will now illustrate the preceding theory of the binomial expansion by several numerical cases. In our first example, for simplicity, we will consider only the binomial coefficients in the expansion of $(q + p)^n$.

Example 1. For the binomial expansion of $(q + p)^8$, draw a graph on which each point has the number of a term as its abscissa and the value of the corresponding binomial coefficient as its ordinate.

SOLUTION. By Sec. 13.7, for $n = 8$, we readily find the nine binomial coefficients, taken in order, to be

$$1,\ 8,\ 28,\ 56,\ 70,\ 56,\ 28,\ 8,\ 1.$$

The coordinates of the points, as shown in Fig. 41, are (1, 1), (2, 8), (3, 28), and so on. A "smooth" curve is drawn through these points. We

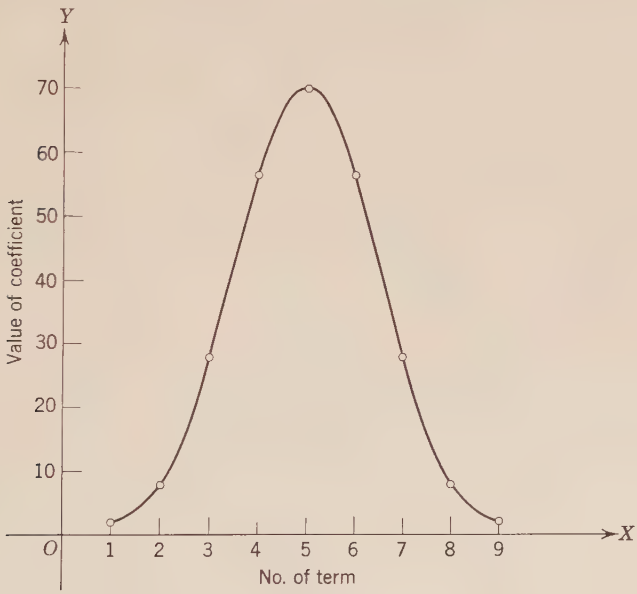


Figure 41

call this curve the *graph* of the coefficients although it is only an approximation, for we have no data on the graph between points. But as n, and therefore the number of terms, increases, the resulting graph approaches more and more closely the shape shown in Fig. 41. This bell shape is typical of *probability curves*, which we will discuss later.

In the next numerical example we will consider the graphical representation of the values of the individual terms, and not just their binomial coefficients, in the expansion of $(q + p)^n$.

Example 2. Find the values of the individual terms in the binomial expansion of $(\frac{2}{5} + \frac{3}{5})^6$, and set up a table with the following six columns of corresponding values:

(1) Number of term in expansion.
(2) Value of r (number of successes).
(3) Probability P_1 of exactly r successes.
(4) Probability P_2 of at least r successes.
(5) Individual frequency.
(6) Cumulative frequency.

Plot two curves, each to have the values in column (2) as abscissas, one curve to have column (3) and the other to have column (4) as ordinates. Compute the most probable value of r and verify it in the table.

SOLUTION. Table 1 shows the required values.

TABLE 1

EXPANSION OF $(\frac{2}{5} + \frac{3}{5})^6$

$n = 6,\ p = \frac{3}{5},\ q = 1 - p = \frac{2}{5}$

No. of Term	r	Probability: Exactly r successes in n trials = value of term	Probability: At least r successes in n trials	Frequency: Individual	Frequency: Cumulative
		$P_1 = C(n, r)p^r q^{n-r}$	$P_2 = \sum_{r=0}^{n} C(n, r)p^r q^{n-r}$	nP_1	nP_2
(1)	(2)	(3)	(4)	(5)	(6)
1	0	0.004096	1.000000	0.024576	6.000000
2	1	0.036864	0.995904	0.221184	5.975424
3	2	0.138240	0.959040	0.829440	5.754240
4	3	0.276480	0.820800	1.658880	4.924800
5	4	0.311040	0.544320	1.866240	3.265920
6	5	0.186624	0.233280	1.119744	1.399680
7	6	0.046656	0.046656	0.279936	0.279936
Totals		1.000000		6.000000	

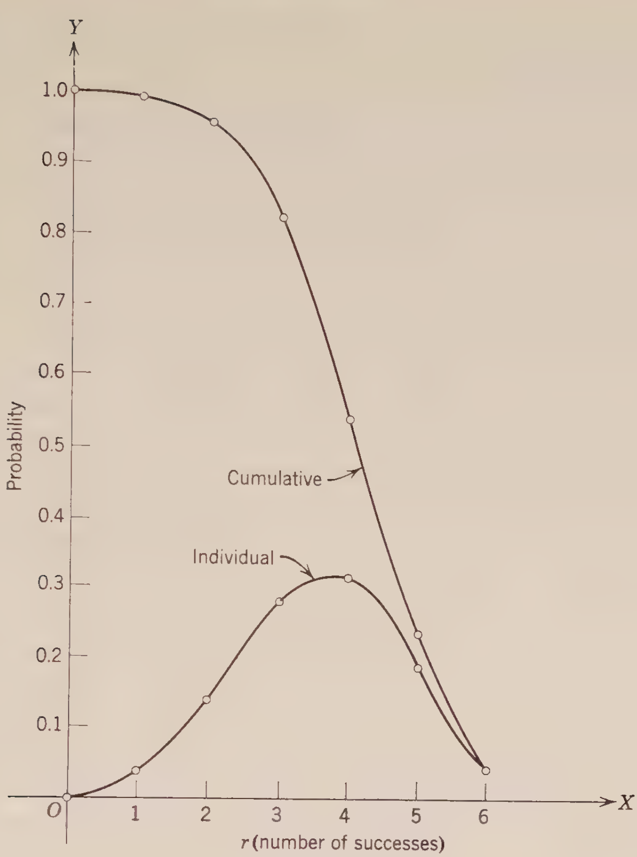


Figure 42

Columns (3) and (4) follow from Theorems 4 and 5, respectively, of Sec. 14.5. Columns (5) and (6) give the frequency or expected number of occurrences in n (= 6) trials (Sec. 14.2). The values in column (5) constitute what is known as an *individual frequency distribution* and the values in column (6) a *cumulative frequency distribution.* Columns (5) and (6) pair off with columns (3) and (4), respectively, (3) and (5) giving individual values and (4) and (6) cumulative values.

It should be noted that the sum of the values in column (3) is unity, the value of certainty. This must necessarily be so, for it represents the sum of the probabilities of all possible cases. A similar remark applies to column (5) where the sum of the values is 6, the total number of trials.

We next plot the values in column (2) as abscissas and the values in columns (3) and (4) as ordinates. The results are the two curves shown in Fig. 42 and known as *probability curves.* We note that the individual curve has approximately the bell shape typical of probability curves. This curve is not symmetrical like the curve in Fig. 41; it is then said to be *skew*. If $p = q = \frac{1}{2}$, however, the individual probability curve is symmetrical. The points on the cumulative curve give the probability of r or more successes.

By Theorem 6, the most probable value of r is given by

$$np + p > r > np - q.$$

For $n = 6$, $p = \frac{3}{5}$, $q = \frac{2}{5}$, $\quad 6 \cdot \frac{3}{5} + \frac{3}{5} > r > 6 \cdot \frac{3}{5} - \frac{2}{5}$

or $\quad 4\frac{1}{5} > r > 3\frac{1}{5}.$

Hence the most probable value of r is 4, and for this value the table gives $P_1 = 0.31104$, the maximum individual probability.

We can also plot the values in columns (5) and (6) but since they are proportional, respectively, to the values in columns (3) and (4), the resulting curves appear similar in shape to those in Fig. 42. They are called, respectively, *individual* and *cumulative frequency curves*.

NOTE. This example is for a very limited range with the number of trials $n = 6$. For larger values of n, the amount of computation increases considerably, but the probability curves exhibit the same basic characteristics. As n increases, the individual graph comes closer and closer to a smooth, bell-shaped curve.

When the observations or values of a set are proportional to the terms of a binomial expansion, they are said to form a *binomial distribution*. There are various types of distributions; among them is the *normal distribution* leading to the well-known *normal probability curve*. These distributions and their corresponding frequency curves are of fundamental importance in the science of statistics, but their discussion is beyond the scope of this book.

EXERCISES. GROUP 53

1. Establish Theorem 5 (Sec. 14.5) by considering the events to happen exactly $r, r + 1, r + 2, \cdots, n$ times, and show that the result is exactly the same as relation (2) except that the summation is in reverse order.

2. A coin is tossed 6 times. Find the probability of obtaining exactly 2 heads.

3. A single die is thrown 6 times. Find the probability of obtaining exactly 5 aces.

4. The probability that A wins a certain game is $\frac{1}{3}$. If 7 of these games are played under exactly the same conditions, find the probability that A wins exactly 4 of them.

5. A coin is tossed n times. Show that the probability of obtaining exactly r tails is $C(n, r) \div 2^n$.

6. From a deck of 52 cards, a single card is drawn at random. The card is then replaced, the deck is thoroughly shuffled, and again a single card is drawn at random and then replaced. This process is performed a total of 5 times. Find the probability of obtaining exactly 3 spades.

7. From a bag containing 3 white and 2 black balls, a single ball is drawn at random. The ball is then replaced, the balls are thoroughly mixed, and again a single ball is drawn at random. This process is performed a total of 4 times. Find the probability of obtaining exactly (a) 2 white balls; (b) 4 black balls.

8. In 6 throws of a pair of dice, find the probability of obtaining exactly three 7's.

9. A baseball player whose batting average is 0.300 comes to bat 4 times in a particular game. Find the probability that he gets exactly 2 hits.

10. On the average, a certain student correctly solves 5 out of every 6 problems that he attempts. In a test consisting of 8 problems, find the probability that he solves exactly 6 correctly.

11. A coin is tossed 10 times. Find the probability of obtaining at least 8 heads.

12. A single die is thrown 7 times. Find the probability of obtaining at least 5 aces.

13. A pair of dice is thrown 5 times. Find the probability of obtaining at least four 7's.

14. The probability of A winning a single game is $\frac{2}{3}$. Find the probability that in a series of 6 games, he will win at least 4 of them.

15. On the average, a marksman hits a target 300 times in 400 shots. Find the probability that he will hit the target at least 3 times in 5 shots.

16. In the manufacture of a certain article, it is found in the long run that 1 per cent of the product is defective. For a random sample of 10 articles, find the probability that not more than 2 are defective.

17. The passing mark for a test consisting of 10 problems is 70 per cent. On the average, a certain student solves correctly 4 out of every 5 problems that he attempts. Find the probability that he will pass the test.

18. It is found by inspection that, on the average, one out of every 50 automobiles has defective headlights. Find the probability that out of 10 automobiles taken at random, at least one will pass the inspection of headlights.

19. The probability that a man 50 years old will be alive 20 years later is 0.6. Out of a group of 5 men aged 50, find the probability that at least 4 will be alive at 70.

20. A and B play a game in which A's skill is to B's skill as 3 is to 2. Find the probability that A wins at least 1 game out of 4.

21. If q is the probability that an event will fail in a single trial, show that the probability of at least one success in n trials is $1 - q^n$.

22. If p is the probability that an event will occur in a single trial, show that the probability of at least one failure in n trials is $1 - p^n$.

23. A coin is tossed 8 times. Find the probability of obtaining an odd number of heads.

24. *A* continues to throw a single die until a 6 appears. Find the probability that he will have to make (a) at least 10 throws; (b) exactly 10 throws.

25. A box contains 6 cards all marked differently. One card is drawn at random and then replaced. The cards are thoroughly shuffled and another card is drawn at random and then replaced. This process is performed a total of 6 times. Find the probability that every card has been drawn.

26. A coin is tossed 8 times. Find the most probable number of heads and the probability of that number.

27. A coin is tossed 10 times. Find the most probable number of tails and the probability of that number.

28. A die is thrown 12 times. Find the most probable number of aces and the probability of that number.

29. The probability of *A* winning a single game is $\frac{1}{3}$. Find the most probable number of his victories in a series of 12 games and the probability of that number.

30. In Theorem 6 (Sec. 14.6), if $np + p$ and $np - q$ are integers, show that r has two values, that is, show that there are two equal terms in the expansion of $(q + p)^n$, each of which is larger than any other term. Verify this for the expansion of $(\frac{1}{2} + \frac{1}{2})^7$.

31. A coin is tossed 9 times. Find the probability of either one of the most probable number of heads.

32. For a single trial of an event, let p be the probability of success and q the probability of failure. If, for n repeated trials, np is an integer, show that the most probable number of failures is equal to nq.

33. As in Example 1 of Sec. 14.6, draw the graph for the binomial coefficients of $(q + p)^{10}$.

34. As in Example 1 of Sec. 14.6, draw the graph for the binomial coefficients of $(q + p)^{15}$.

35. Plot the individual and cumulative frequency curves for the binomial expansion of Example 2 of Sec. 14.6.

In each of Exs. 36–40, for the given binomial expansion, as in Example 2 of Sec. 14.6, make up a similar table of computations, plot the same types of curves, and make the same kind of analysis.

36. $(\frac{1}{2} + \frac{1}{2})^6$. 37. $(\frac{5}{6} + \frac{1}{6})^6$. 38. $(\frac{3}{4} + \frac{1}{4})^{10}$. 39. $(\frac{2}{3} + \frac{1}{3})^9$.
40. $(\frac{7}{10} + \frac{3}{10})^{15}$.

15

Determinants

15.1. INTRODUCTION

The subject of determinants has been studied extensively over a considerable period of time. Although the concept of a determinant had its origin in the solution of a system of linear equations, it has since been used in a great variety of situations. Thus, as will be noted in several exercises of this chapter, the equations of certain geometric curves and figures may be written in determinant form. Also, there are numerous instances where a property or relation depends upon the value of a special determinant. Furthermore, determinants are useful in the study of matrices, which, as we have previously noted (Sec. 1.6) are important in modern mathematics and physics.

The principles and evaluation of determinants are fairly easy to comprehend. Such difficulties that the student may encounter in his introduction to the subject are due mainly to the fact that he must learn some new rules of procedure. The arrangement of the material of this chapter takes this fact into account. Accordingly, we will start by illustrating both principles and operations when applied to the simplest forms of a determinant.

15.2. NATURE OF A DETERMINANT

It is necessary at the very outset for the student to have some idea of the form and nature of a determinant. We therefore state that a *determinant of order n*, designated by Δ_n, is represented, as shown in (1), by a square array of n^2 quantities, called *elements*, arranged in n *rows* and n *columns.*

(1)
$$\Delta_n = \begin{vmatrix} a_1 & b_1 & \cdots & l_1 \\ a_2 & b_2 & \cdots & l_2 \\ \cdots & \cdots & \cdots & \cdots \\ a_n & b_n & \cdots & l_n \end{vmatrix}$$

It is customary to enclose this array by two vertical lines.

For convenience, we refer to the rows and columns by numbers. Thus, the first row consists of the n elements $a_1, b_1, \cdots, l_1$, the second row of the n elements $a_2, b_2, \cdots, l_2$, and so on. Similarly, the first column consists of the n elements $a_1, a_2, \cdots, a_n$, the second column of the n elements $b_1, b_2, \cdots, b_n$, and so on. It must be emphasized that we have *not* defined the term determinant here; we have merely given a description of its appearance and not its value. Although we will give a precise definition in a later section, it will be sufficient at this point to state that a determinant is equal to the algebraic sum of terms, each of which is the product of n elements, one and only one from each row and from each column.

Since n represents the *order* of a determinant, it follows that a determinant of order 2 has 2 rows and 2 columns, a determinant of order 3 has 3 rows and 3 columns, and so on. Hence the determinant of lowest order is obtained for $n = 1$ and may be represented by $|a_1|$. It has simply one element, one row, and one column, and its value is *defined* as the element itself, that is, $|a_1| = a_1$. We will, in general, consider only determinants of order $n \geq 2$.

NOTE. The student should be careful not to confuse the vertical lines used as the symbol of a determinant with the vertical lines used to designate the absolute value of a quantity (Sec. 2.4). Thus, as an absolute value, $|-4| = 4$, but as a determinant, $|-4| = -4$.

15.3. DETERMINANTS OF ORDER 2

We exhibit the determinant of order 2 as

$$\Delta_2 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix},$$

where the elements a_1 and b_2 are said to lie along the *principal diagonal.* The value of Δ_2 is *defined* as the product of the elements on the principal diagonal minus the product of the elements on the other diagonal. That is, by definition,

$$(1)\qquad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1,$$

where the right member is called the *expansion* of Δ_2.

Thus, as a numerical illustration, we have

$$\begin{vmatrix} 2 & 3 \\ -4 & 1 \end{vmatrix} = 2 \cdot 1 - (-4 \cdot 3) = 2 + 12 = 14.$$

We will now show how determinants of order 2 are associated with the solution of a system of 2 linear equations in 2 variables. In Sec. 4.7 we established Theorem 2, part of which we repeat here for convenience.

The linear system

$$(2)\qquad \begin{aligned} a_1x + b_1y &= c_1, \\ a_2x + b_2y &= c_2, \end{aligned}$$

has the unique solution

$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}, \qquad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

if, and only if, $a_1b_2 - a_2b_1 \neq 0$.

Now, in view of our definition of a determinant of order 2, the solution of the system (2) may be written in determinant form as follows:

$$(3)\qquad x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0.$$

We observe that each value in the solution has a common denominator, which is called the *determinant of the system*. Also, for the value of x, the numerator is obtained from the denominator by replacing the *first* column of coefficients a_1 and a_2 by the constant terms c_1 and c_2, respectively. Similarly, for the value of y, the numerator is obtained from the denominator by replacing the *second* column of coefficients b_1 and b_2 by the constant terms c_1 and c_2, respectively.

NOTES. 1. It is evident that if one or more elements of a determinant are interchanged, the value of the determinant may be changed. Hence, in using the determinant form of solution (3), it is highly important to form the columns of coefficients in the correct *order*. For this reason, the system (2) should always be written so that the same variables appear under each other and the constant terms are on the right-hand side. If a variable is absent, its coefficient is taken as zero.

2. The determinant solution (3) is known as *Cramer's rule.* We will see later that this rule applies to the general case of a system of n linear equations in n variables, where n is any positive integer.

As an illustration of Cramer's rule we have the following

Example. Using determinants only, solve the system

$$2x + 3y + 1 = 0,$$
$$2y - 3x = 8.$$

SOLUTION. Ir accordance with Note 1 above, we rewrite the given system in the form

$$2x + 3y = -1,$$
$$3x - 2y = -8.$$

Our next step is to evaluate the determinant Δ of the system, for we have a unique solution if, and only if, $\Delta \neq 0$. We find here that

$$\Delta = \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} = 2(-2) - 3 \cdot 3 = -4 - 9 = -13.$$

Then, by Cramer's rule, we have

$$x = \frac{\begin{vmatrix} -1 & 3 \\ -8 & -2 \end{vmatrix}}{\Delta} = \frac{2 + 24}{-13} = -2,$$

$$y = \frac{\begin{vmatrix} 2 & -1 \\ 3 & -8 \end{vmatrix}}{\Delta} = \frac{-16 + 3}{-13} = 1.$$

At this point it will be advantageous to discuss some of the properties of determinants, for they may be illustrated very simply by means of determinants of order 2. Later these properties will be presented as theorems which hold for determinants of *any* order.

PROPERTY 1. If the corresponding rows and columns of a determinant are interchanged, the value of the determinant remains unchanged.

Thus, $$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1,$$

and $$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - b_1a_2.$$

In view of this property, it follows that any theorem on determinants which is true for its rows is also true for its columns.

PROPERTY 2. If every element of any row (or column) is zero, the value of the determinant is zero.

Thus, $$\begin{vmatrix} 0 & 0 \\ b_1 & b_2 \end{vmatrix} = 0(b_2) - b_1(0) = 0.$$

PROPERTY 3. If two rows (or columns) of a determinant are interchanged, the value of the determinant is changed in sign but not numerically.

Thus, $$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1,$$

and $$\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix} = a_2b_1 - a_1b_2 = -\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

PROPERTY 4. If the corresponding elements of two rows (or columns) of a determinant are equal, the value of the determinant is zero.

Thus, $$\begin{vmatrix} a_1 & b_1 \\ a_1 & b_1 \end{vmatrix} = a_1b_1 - a_1b_1 = 0.$$

PROPERTY 5. If each element of a row (or column) of a determinant is multiplied by the same number k, the new determinant has a value equal to k times that of the original determinant.

Thus, $$\begin{vmatrix} ka_1 & kb_1 \\ a_2 & b_2 \end{vmatrix} = ka_1b_2 - a_2kb_1$$
$$= k(a_1b_2 - a_2b_1) = k\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

PROPERTY 6. If each element of some row (or column) of a determinant is the sum of two quantities, this determinant may be written as the sum of two determinants, that is,

$$\begin{vmatrix} a_1 + a_1' & b_1 \\ a_2 + a_2' & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} a_1' & b_1 \\ a_2' & b_2 \end{vmatrix}.$$

Thus, $$\begin{vmatrix} a_1 + a_1' & b_1 \\ a_2 + a_2' & b_2 \end{vmatrix} = a_1b_2 + a_1'b_2 - a_2b_1 - a_2'b_1$$
$$= (a_1b_2 - a_2b_1) + (a_1'b_2 - a_2'b_1)$$
$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} a_1' & b_1 \\ a_2' & b_2 \end{vmatrix}.$$

PROPERTY 7. If each element of any row (or column) of a determinant is multiplied by the same number k and added to the corresponding element of another row (or column), the value of the determinant remains unchanged. That is,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 + kb_1 & b_1 \\ a_2 + kb_2 & b_2 \end{vmatrix}.$$

Thus,

$$\begin{vmatrix} a_1 + kb_1 & b_1 \\ a_2 + kb_2 & b_2 \end{vmatrix} = a_1b_2 + kb_1b_2 - (a_2b_1 + kb_1b_2)$$

$$= a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

EXERCISES. GROUP 54

In each of Exs. 1–7, evaluate the given determinant.

1. $\begin{vmatrix} 3 & 4 \\ 5 & 2 \end{vmatrix}$. 2. $\begin{vmatrix} -2 & 1 \\ 7 & -4 \end{vmatrix}$. 3. $\begin{vmatrix} 2 & 6 \\ 4 & -2 \end{vmatrix}$. 4. $\begin{vmatrix} 3 & 1 \\ -3 & -1 \end{vmatrix}$.

5. $\begin{vmatrix} x & 2a \\ 2x & a \end{vmatrix}$. 6. $\begin{vmatrix} 1 & 2 \\ y^2 & x^2 \end{vmatrix}$. 7. $\begin{vmatrix} x+1 & 2 \\ 2x & x-3 \end{vmatrix}$.

In each of Exs. 8–9, solve the given equation for x.

8. $\begin{vmatrix} 3 & 2 \\ x-2 & x \end{vmatrix} = 0.$ 9. $\begin{vmatrix} x & x+6 \\ 1 & x+2 \end{vmatrix} = 0.$

In each of Exs. 10–15, solve the given system, using determinants only.

10. $2x - 3y = 5$, $3x + 2y = 1$.
11. $2x + 3y = 4$, $x - y = 7$.
12. $4x - y = 11$, $y + 2x = 1$.
13. $2x + 3y = 6$, $x - y + 7 = 0$.
14. $3x + 2y = 0$, $3y - 2x = 0$.
15. $x + 2y = 5$, $2x + 4y = 3$.
16. Establish Property 4 (Sec. 15.3) by using Property 3.
17. By means of Property 5 (Sec. 15.3), show that if all the elements of any row (or column) of a determinant have a common factor, the expansion of the determinant also has that factor.
18. Show that $k\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} ka_1 & b_1 \\ ka_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & kb_1 \\ a_2 & kb_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_2 \\ ka_1 & kb_2 \end{vmatrix}.$
19. Show that $\begin{vmatrix} a_1 & b_1 + b_1' \\ a_2 & b_2 + b_2' \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} a_1 & b_1' \\ a_2 & b_2' \end{vmatrix}.$
20. As an extension of Property 6 (Sec. 15.3), show that

$$\begin{vmatrix} a_1 + a_1' + a_1'' & b_1 \\ a_2 + a_2' + a_2'' & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} a_1' & b_1 \\ a_2' & b_2 \end{vmatrix} + \begin{vmatrix} a_1'' & b_1 \\ a_2'' & b_2 \end{vmatrix}.$$

21. Establish Property 7 (Sec. 15.3) by showing that

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 + ka_2 & b_1 + kb_2 \\ a_2 & b_2 \end{vmatrix}.$$

22. Establish Property 7 (Sec. 15.3) by using Properties 6, 5, and 4.

23. Verify Property 7 (Sec. 15.3) by means of numerical examples.

24. Show that $\begin{vmatrix} a_1 + a_1' & b_1 + b_1' \\ a_2 + a_2' & b_2 + b_2' \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} a_1' & b_1 \\ a_2' & b_2 \end{vmatrix}$

$$+ \begin{vmatrix} a_1 & b_1' \\ a_2 & b_2' \end{vmatrix} + \begin{vmatrix} a_1' & b_1' \\ a_2' & b_2' \end{vmatrix}.$$

25. Show that $\begin{vmatrix} a_1 + b_1 & b_1 + b_1' \\ a_2 + b_2 & b_2 + b_2' \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} a_1 & b_1' \\ a_2 & b_2' \end{vmatrix} + \begin{vmatrix} b_1 & b_1' \\ b_2 & b_2' \end{vmatrix}.$

15.4. DETERMINANTS OF ORDER 3

We now go one step further and study the determinant of order 3, which we exhibit as

$$(1) \qquad \Delta_3 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and *define* by the expansion

$$(2) \qquad \Delta_3 = a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1.$$

The expansion (2) may, of course, be used as a formula for evaluating any determinant of order 3. However, this formula is not convenient in evaluating a determinant with numerical elements, for, in substituting, we must be careful to assign each element its proper row and column. For this reason there are in common use two arrangements whereby it is possible to obtain the terms of the expansion as the products of elements along certain diagonals. However, since these two devices cannot be used for determinants of order higher than 3, we will not consider them here. Instead we shall use a method applicable to a determinant of any order and, because it is the most convenient method, we shall employ it hereafter.

The basic idea in this method of evaluation is to express the expansion of a given determinant in terms of determinants of lower order. Thus, by expressing a determinant of order 3 in terms of determinants of order 2, we can readily obtain its value, for determinants of order 2 are easily computed. This method of expansion is known as *development by minors.*

Definition. The *minor* of an element of a determinant is the determinant of next lower order obtained by deleting the row and column in which that element lies.

Thus, for Δ_3 as given in (1), the minor of the element b_1 is obtained by striking out the first row and second column in which b_1 lies. The minor is then $\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$, a determinant of order 2. Similarly, the minor of c_2 is $\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$, and so on.

Closely related to the term minor is another term for which we have the

Definition. The *cofactor* of an element of a determinant is equal to the minor of that element preceded by a plus or minus sign depending on whether the sum of the number of the row and the number of the column in which that element lies is even or odd.

Thus, for Δ_3, the cofactor of the element c_1 in the first row and third column is $\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$ since $1 + 3 = 4$, an even number. Similarly, the cofactor of the element a_2 in the second row and first column is $-\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$ since $2 + 1 = 3$, an odd number.

At this time we will state *without proof* an important theorem which we shall use hereafter for evaluating any determinant.

Theorem. *The value of any determinant of order n is equal to the sum of n products each of which is formed by multiplying each element of any one row (or column) by its corresponding cofactor.*

The determinant is then said to be *expanded with respect to the elements of that particular row (or column).*

It is easy to verify this theorem for Δ_3. Thus, expanding Δ_3 with respect to the elements of the first row, we have

$$\Delta_3 = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1,$$

which agrees with the expansion (2) above.

Note that the theorem states that this expansion may be made with respect to the elements of *any one* row (or column). Thus, expanding Δ_3

with respect to the elements of the second column, we have

$$\Delta_3 = -b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

$$= -a_2b_1c_3 + a_3b_1c_2 + a_1b_2c_3 - a_3b_2c_1 - a_1b_3c_2 + a_2b_3c_1,$$

which also agrees with the expansion (2) above.

We will next illustrate the theorem by a numerical example, but, before doing so, we call attention to

NOTE 1. For neatness and compactness, negative elements in a determinant will hereafter be written with the minus sign *above* instead of in front of the element.

Example 1. Evaluate the following determinant by expansion with respect to the elements of (a) the third row; (b) the second column:

$$\Delta_3 = \begin{vmatrix} 1 & 4 & \bar{2} \\ \bar{3} & 1 & 0 \\ 5 & \bar{2} & 3 \end{vmatrix}.$$

SOLUTION. (a) $\Delta_3 = 5 \begin{vmatrix} 4 & \bar{2} \\ 1 & 0 \end{vmatrix} - (-2) \begin{vmatrix} 1 & \bar{2} \\ \bar{3} & 0 \end{vmatrix} + 3 \begin{vmatrix} 1 & 4 \\ \bar{3} & 1 \end{vmatrix}$

$$= 10 - 12 + 3 + 36 = 37.$$

(b) $\Delta_3 = -4 \begin{vmatrix} \bar{3} & 0 \\ 5 & 3 \end{vmatrix} + 1 \begin{vmatrix} 1 & \bar{2} \\ 5 & 3 \end{vmatrix} - (-2) \begin{vmatrix} 1 & \bar{2} \\ \bar{3} & 0 \end{vmatrix}$

$$= 36 + 3 + 10 - 12 = 37.$$

When the theorem is applied to a determinant of high order, it is evident that the complete expansion entails a considerable amount of arithmetical computation. We now make the important observation that if a particular row (or column) has one or more zeros, this computation is considerably reduced by expanding with respect to that row (or column). Moreover, it is possible to create such zeros without changing the value of the determinant by using Property 7 (Sec. 15.3). We illustrate the process in

Example 2. By creating as many zeros as possible in some one row or column, evaluate the determinant

$$\Delta_3 = \begin{vmatrix} 2 & 1 & \bar{3} \\ 4 & \bar{2} & 5 \\ 3 & 2 & \bar{7} \end{vmatrix}.$$

SOLUTION. Property 7 (Sec. 15.3) states that if each element of any row (or column) of a determinant is multiplied by the same number k and added to the corresponding element of another row (or column), the value of the determinant remains unchanged. Thus we can create one zero element in the first row and first column by multiplying each element of the second column by -2 and adding the result to the corresponding element of the first column. This gives us

$$\text{(3)} \qquad \Delta_3 = \begin{vmatrix} 2-2 & 1 & \bar{3} \\ 4+4 & \bar{2} & 5 \\ 3-4 & 2 & \bar{7} \end{vmatrix} = \begin{vmatrix} 0 & 1 & \bar{3} \\ 8 & \bar{2} & 5 \\ \bar{1} & 2 & \bar{7} \end{vmatrix}.$$

We can now create another zero element in the first row and third column by multiplying each element of the second column by 3 and adding the result to the corresponding element of the third column. Then, from (3), we have

$$\Delta_3 = \begin{vmatrix} 0 & 1 & \bar{3}+3 \\ 8 & \bar{2} & 5-6 \\ \bar{1} & 2 & \bar{7}+6 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 8 & \bar{2} & \bar{1} \\ \bar{1} & 2 & \bar{1} \end{vmatrix}.$$

We have shown these operations in two steps for clarity, but since the operating column is the same (the second) in both steps, we can show the result in one step. Furthermore, the arithmetical work can be performed mentally and the results put down immediately. We will also hereafter indicate the operating column (or row) by an asterisk. Our work will now appear compactly as follows:

$$\text{(4)} \qquad \Delta_3 = \begin{vmatrix} 2 & \overset{*}{1} & \bar{3} \\ 4 & \bar{2} & 5 \\ 3 & 2 & \bar{7} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 8 & \bar{2} & \bar{1} \\ \bar{1} & 2 & \bar{1} \end{vmatrix}.$$

By expanding with respect to the elements of the first row in (4), we have only one minor to compute, that is,

$$\Delta_3 = -\begin{vmatrix} 8 & \bar{1} \\ \bar{1} & \bar{1} \end{vmatrix} = -(-8-1) = 9.$$

In general, by using Property 7 (Sec. 15.3), it is possible to transform any given determinant into another of the same value but having all zero elements, except one, in some row (or column). By expanding this new determinant with respect to the elements of that row (or column), we obtain a single determinant of the next lower order. Note that if the use of Property 7 produces *all* zero elements in some row (or column), the given determinant is equal to zero by Property 2 (Sec. 15.3).

Since this method is efficient for evaluating any determinant, and is therefore the method we shall use hereafter, we set up the procedure here for future reference.

Procedure for evaluating any determinant

1. Select an operating row (or column) in the given determinant and indicate it by an asterisk.
2. In accordance with Property 7 (Sec. 15.3), multiply each element of the operating row (or column) by such a number that, when added to the corresponding element of another row (or column), at least one zero element is obtained.
3. Repeat Step 2 as often as necessary in order to obtain an equivalent determinant in which all the elements, except one, of some row (or column) are zero.
4. Expand the determinant obtained in Step 3 with respect to the row (or column) having all zero elements except one, thus obtaining a single determinant of the next lower order.
5. Repeat the preceding process for the determinant obtained in Step 4.
6. Continue this procedure until a determinant of order 2 is obtained, and evaluate it.

We will illustrate the above procedure by applying it to a determinant of order 4. But before doing so, we call attention to

NOTE 2. The creation of zero elements by the use of Property 7 is simple provided that we have a unit element in the operating row (or column), otherwise the process involves fractions and cumbersome arithmetic. But in such cases a preliminary use of Property 7 can produce the required unit element, as the following example illustrates.

Example 3. Evaluate the determinant

$$\Delta_4 = \begin{vmatrix} \bar{2} & 3 & \bar{5} & 2 \\ 5 & \bar{2} & 7 & 3 \\ 4 & \bar{3} & 6 & 5 \\ \bar{3} & 2 & 2 & 4 \end{vmatrix}.$$

SOLUTION. The various steps in the evaluation are exhibited first; the explanation follows.

$$\Delta_4 = \begin{vmatrix} \bar{2} & 3 & \bar{5} & 2 \\ 5 & \bar{2} & 7 & 3 \\ 4 & \bar{3} & 6 & 5 \\ \bar{3} & 2 & 2 & 4 \end{vmatrix} \begin{matrix} \\ * \\ \\ \\ \end{matrix} = \begin{vmatrix} 3 & \overset{*}{1} & 2 & 5 \\ 5 & \bar{2} & 7 & 3 \\ 4 & \bar{3} & 6 & 5 \\ \bar{3} & 2 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 11 & \bar{2} & 11 & 13 \\ 13 & \bar{3} & 12 & 20 \\ \bar{9} & 2 & \bar{2} & \bar{6} \end{vmatrix}$$

$$= - \begin{vmatrix} 11 & \overset{*}{11} & 13 \\ 13 & 12 & 20 \\ \bar{9} & \bar{2} & \bar{6} \end{vmatrix} = - \begin{vmatrix} 0 & 11 & 13 \\ 1 & 12 & 20 \\ \bar{7} & \bar{2} & \bar{6} \end{vmatrix} \begin{matrix} \\ * \\ \\ \end{matrix} = - \begin{vmatrix} 0 & 11 & 13 \\ 1 & 12 & 20 \\ 0 & 82 & 134 \end{vmatrix}$$

$$= \begin{vmatrix} 11 & 13 \\ 82 & 134 \end{vmatrix} = 2 \begin{vmatrix} 11 & 13 \\ 41 & 67 \end{vmatrix} = 2(737 - 533) = 408.$$

EXPLANATION. The given determinant has no unit element. But by adding the second row (indicated by an asterisk) to the first row, we obtain a unit element in the first row and second column.

Using Property 7 with the second column as an operating column (indicated by an asterisk), we obtain 3 zero elements in the first row.

Expanding with respect to the elements of the first row, we obtain a single determinant of order 3. Since this determinant has no unit element, we subtract the second column (indicated by an asterisk) from the first column. This gives us a unit element in the second row and first column. Then by adding 7 times the elements of the second row (indicated by an asterisk) to the corresponding elements of the third row, we obtain a determinant of order 3 with two zero elements in the first column.

Expanding this last determinant of order 3 with respect to the elements of the first column, we obtain a single determinant of order 2, which is readily evaluated as shown.

Since this section is primarily concerned with determinants of order 3, we will now consider the solution of a system of 3 linear equations in 3 variables:

$$\begin{aligned} a_1x + b_1y + c_1z &= k_1, \\ a_2x + b_2y + c_2z &= k_2, \\ a_3x + b_3y + c_3z &= k_3. \end{aligned} \tag{5}$$

The solution of this system may be effected by the method of elimination studied in Sec. 4.7 and is therefore left as an exercise to the student.

By using determinants, we may write this solution in the form

$$(6)\qquad x = \frac{\begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}}{\Delta_3}, \quad y = \frac{\begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}}{\Delta_3}, \quad z = \frac{\begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}}{\Delta_3},$$

where Δ_3, the determinant of the system (5), is given by

$$\Delta_3 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0.$$

By evaluating these determinants, the student should show that the determinant solution (6) is precisely the same as that obtained by the method of elimination. We also see here the motivation for the definition of Δ_3 as given at the beginning of this section.

It should be observed that the determinant solution (6) is analogous to the determinant solution (3) of the system (2) of two linear equations studied in Sec. 15.3. This solution is, of course, another example of Cramer's rule.

Example 4. Using determinants only, solve the system

$$\begin{aligned} 3x + 2y - z &= 3, \\ 4x - y - 3z &= 0, \\ x - 2y - 3z &= 1. \end{aligned}$$

SOLUTION. The actual evaluation of all the determinants involved is left as an exercise to the student.

The determinant Δ of the system is

$$\begin{vmatrix} 3 & 2 & \bar{1} \\ 4 & \bar{1} & \bar{3} \\ 1 & \bar{2} & \bar{3} \end{vmatrix} = 16 \neq 0.$$

Hence, by Cramer's rule, the solution is

$$x = \frac{\begin{vmatrix} 3 & 2 & \bar{1} \\ 0 & \bar{1} & \bar{3} \\ 1 & \bar{2} & \bar{3} \end{vmatrix}}{\Delta} = \frac{-16}{16} = -1,$$

$$y = \frac{\begin{vmatrix} 3 & 3 & \bar{1} \\ 4 & 0 & \bar{3} \\ 1 & 1 & \bar{3} \end{vmatrix}}{\Delta} = \frac{32}{16} = 2,$$

$$z = \frac{\begin{vmatrix} 3 & 2 & 3 \\ 4 & \bar{1} & 0 \\ 1 & \bar{2} & 1 \end{vmatrix}}{\Delta} = \frac{-32}{16} = -2.$$

EXERCISES. GROUP 55

In each of Ex. 1–8, evaluate the given determinant.

1. $\begin{vmatrix} 2 & 1 & \bar{1} \\ 3 & 2 & 5 \\ 4 & \bar{3} & 6 \end{vmatrix}.$ 2. $\begin{vmatrix} 3 & 6 & 9 \\ 2 & 4 & 6 \\ 7 & \bar{9} & 5 \end{vmatrix}.$ 3. $\begin{vmatrix} 4 & \bar{3} & 3 \\ 2 & 4 & 3 \\ \bar{5} & 7 & 2 \end{vmatrix}.$

4. $\begin{vmatrix} 2 & 3 & 4 \\ 4 & 6 & 7 \\ \bar{1} & 2 & 5 \end{vmatrix}.$ 5. $\begin{vmatrix} 2 & 3 & 1 \\ 8 & 4 & 3 \\ \bar{2} & 5 & \bar{1} \end{vmatrix}.$ 6. $\begin{vmatrix} 3 & 2 & 5 \\ 7 & 3 & 7 \\ 15 & 6 & 8 \end{vmatrix}.$

7. $\begin{vmatrix} 2 & 3 & 2 & 3 \\ 3 & 6 & 1 & 5 \\ 7 & 14 & 3 & 5 \\ 6 & 12 & 5 & 4 \end{vmatrix}.$ 8. $\begin{vmatrix} 5 & 2 & 7 & 5 \\ 11 & 5 & 8 & 9 \\ 10 & 5 & 3 & 8 \\ 6 & 3 & 2 & 5 \end{vmatrix}.$

In each of Exs. 9–10, solve the given equation for x.

9. $\begin{vmatrix} 2 & 1 & 1 \\ 6 & x & 3 \\ 4 & 2 & x \end{vmatrix} = 0.$ 10. $\begin{vmatrix} \bar{1} & 2 & \bar{3} \\ 2 & x & 6 \\ x & 5 & 2 \end{vmatrix} = 0.$

In each of Exs. 11–15, solve the given system, using determinants only.

11. $x + 2y - z = 3,\ 2x - y + z = 7,\ 2x + y - 4z = -1.$

12. $2x + 7y - 4z = 4$, $x - 3y - 4z = 0$, $2x + 3y + z = 9$.

13. $3x - y - 2z = 4$, $2x + y + 4z = 2$, $7x - 2y - z = 4$.

14. $2x - 3y = 13$, $2y + z = 1$, $x - 2z = -1$.

15. $3x - 9y + 4z = 0$, $5x + 2y - 8z = 0$, $7x - 2y - 5z = 0$.

16. Let C_{ij} and M_{ij} be the cofactor and minor, respectively, of the element a_{ij} lying in the ith row and jth column of a determinant. Show that $C_{ij} = (-1)^{i+j}M_{ij}$.

17. Expand Δ_3 with respect to the elements of the third row and show that the result agrees with the expansion (2) of Sec. 15.4.

18. Verify the theorem of Sec. 15.4 by expanding the determinant of order 2 with respect to the elements of the first column.

19. Solve Example 2 (Sec. 15.4) by using the first row as an operating row.

20. By the method of elimination, find the solution of the system (5) of 3 linear equations given in Sec. 15.4.

21. By evaluating the determinants in the solution (6) of the system (5) in Sec. 15.4, show that the solution is precisely the same as that obtained in Ex. 20.

22. By evaluating all of the determinants involved, check the solution of Example 4 of Sec. 15.4.

In each of Exs. 23–29, verify the specified property for the general determinant Δ_3 of order 3 as given by relation (1) of Sec. 15.4.

23. Property 1 (Sec. 15.3).

24. Property 2 (Sec. 15.3).

25. Property 3 (Sec. 15.3).

26. Property 4 (Sec. 15.3). Use Property 3 (Sec. 15.3).

27. Property 4 (Sec. 15.3). Use Properties 7 and 2 (Sec. 15.3).

28. Property 5 (Sec. 15.3).

29. Property 7 (Sec. 15.3) by showing that

$$\begin{vmatrix} a_1 + kb_1 & b_1 & c_1 \\ a_2 + kb_2 & b_2 & c_2 \\ a_3 + kb_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

30. Show that $\begin{vmatrix} a_1 + a_1' & b_1 & c_1 \\ a_2 + a_2' & b_2 & c_2 \\ a_3 + a_3' & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1' & b_1 & c_1 \\ a_2' & b_2 & c_2 \\ a_3' & b_3 & c_3 \end{vmatrix}.$

31. Show that

$$\begin{vmatrix} a_1 + a_1' + a_1'' & b_1 & c_1 \\ a_2 + a_2' + a_2'' & b_2 & c_2 \\ a_3 + a_3' + a_3'' & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1' & b_1 & c_1 \\ a_2' & b_2 & c_2 \\ a_3' & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1'' & b_1 & c_1 \\ a_2'' & b_2 & c_2 \\ a_3'' & b_3 & c_3 \end{vmatrix}.$$

32. It is shown in analytic geometry that the equation of the straight line passing through the two given distinct points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ may be

written in the form

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

Verify this result by showing (1) that the coordinates of each of the points P_1 and P_2 satisfy the equation and (2) that the expansion of the determinant is linear in the variables x and y.

In each of Exs. 33–34, using the result of Ex. 32, find the equation of the straight line through the two given points.

33. (2, 0), (0, −1).

34. (3, 1), (−2, −1).

35. It is shown in analytic geometry that the area K of the triangle having the vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) is given by

$$K = \tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix},$$

where the absolute value of the determinant is to be taken. Hence, find the area of the triangle whose vertices are (−1, 1), (3, 4), (5, −1).

36. Using the result of Ex. 35, show that a necessary and sufficient condition that three distinct points with coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3) be collinear is that

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

37. Show that $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x - y)(y - z)(z - x).$

38. Show that $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y + z & z + x & x + y \end{vmatrix} = 0.$

39. Show that $\begin{vmatrix} x - y - z & 2x & 2x \\ 2y & y - x - z & 2y \\ 2z & 2z & z - x - y \end{vmatrix} = (x + y + z)^3.$

40. If ω is one of the complex cube roots of unity, evaluate

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}.$$

15.5. DETERMINANTS OF ANY ORDER

We shall now study determinants of any order and show that they have the same properties previously stated and verified for determinants of orders 2 and 3. For this purpose we shall first formulate a definition for a determinant of any order n which will include determinants of orders 2 and 3 as special cases.

Specifically, let us first consider the determinant of order 3:

$$\Delta_3 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

which was previously defined (Sec. 15.4) by the expansion

(1) $\qquad \Delta_3 = a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1.$

Each term in the expansion is the product of three letters which we shall always write in alphabetical order; this is known as their *natural order.* The terms therefore differ from each other only in the *order* of the subscripts 1, 2, 3, which may be permuted in $3! = 6$ different orders (Corollary, Theorem 2, Sec. 13.3). The subscripts in the first term of the expansion are 1, 2, 3, arranged in the order of their magnitude; this is called their *normal order.* But whenever a larger subscript precedes a smaller one, they are said to form an *inversion,* Thus, in the term $a_3b_1c_2$, with the subscripts in the order 312, there are two inversions: 3 preceding 1 and 3 preceding 2. For the term $a_3b_2c_1$, with the subscripts in the order 321, there are three inversions: 3 preceding 2, 3 preceding 1, and 2 preceding 1. The first term $a_1b_2c_3$, whose elements lie along the principal diagonal, has no inversions.

With this understanding of an inversion, we are now ready to give a complete definition of a determinant of any order as follows:

Definition. A determinant of *order n*, where n is any positive integer, is a square array of n^2 quantities called *elements* arranged in n columns and n rows. It represents the algebraic sum of all possible different products, each of n different elements, which can be formed by taking one element and only one element from each column and from each row. A product is preceded by a plus or minus sign depending on whether it presents an even or an odd number of inversions. The single product lying along the principal diagonal and presenting no inversions is preceded by a plus sign; it is called the *leading term.*

NOTES. 1. It should be observed that the sign preceding a term due to its inversions is independent of the sign of the term due to its factors. Thus, if a term presenting an even number of inversions has as its factors the elements 3, -4, and 2, the term itself is equal to $+(3)(-4)(2) = -24$.

2. The student may now easily verify that the definitions of determinants of orders 2 and 3, as given by relation (1) of Sec. 15.3 and relation (2) of Sec. 15.4, respectively, are in accordance with the complete definition of a determinant of any order as given above.

We will first establish the following important theorem on inversions.

Theorem 1. *If any two subscripts are interchanged in any term of the expansion of a determinant, the number of inversions is changed by an odd number, and the sign of the term is changed.*

PROOF. First consider that we interchange two adjacent subscripts. Then the number of inversions is either increased by 1 or decreased by 1, an odd number. Hence, if the original number of inversions is even (a positive term), the interchange produces an odd number of inversions and a negative term, that is, a change in sign. Similarly, if the original number of inversions is odd (a negative term), the interchange produces an even number of inversions and a positive term, again a change in sign.

Next, consider that we interchange two subscripts which are not adjacent but have k numbers between them. To bring the first subscript into the position of the second requires $k + 1$ successive interchanges with adjacent numbers, and this must be followed by k additional successive interchanges with adjacent subscripts to bring the second subscript into the original position of the first, a total of $2k + 1$ interchanges, an odd number. But from above, each interchange with an adjacent subscript changes the number of inversions by 1 or -1 and causes a change in sign. Hence, $2k + 1$ interchanges changes the number of inversions by an odd number, and the sign of the term is changed.

We will now develop some of the properties of a determinant of any order n which, by means of the n letters $a, b, c, \cdots, l$, we write in the form

$$\Delta_n = \begin{vmatrix} a_1 & b_1 & c_1 & \cdots & l_1 \\ a_2 & b_2 & c_2 & \cdots & l_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n & b_n & c_n & \cdots & l_n \end{vmatrix} \tag{2}$$

where the letter denotes the column and the subscript the row in which each element lies.

The leading term in the expansion of Δ_n is the product $a_1b_2c_3 \cdots l_n$. In accordance with the definition of Δ_n, all of the terms of its expansion may be obtained from the leading term by permuting the n subscripts $1, 2, 3, \cdots, n$. This may be done in $n!$ different ways; hence there are $n!$ different terms in the expansion. For $n \geq 2$, $n!$ is an even number.

Now consider any two subscripts. Among the $n!$ different permutations of the subscripts, the first subscript precedes the second as many times as the second precedes the first. But by Theorem 1, the interchange of two subscripts changes the sign of the term. Accordingly, half of the $n!$ terms are preceded by a positive sign and half by a negative sign. We record these results as

Theorem 2. *There are $n!$ different terms in the expansion of a determinant of order n: half of these are preceded by a positive sign and half by a negative sign.*

The properties of a determinant described and illustrated for a determinant of order 2 will now be established as theorems for determinants of *any* order. The student will find it helpful to fix his ideas by illustrating each step in a proof by means of Δ_3, the general determinant of order 3.

Theorem 3. *If the corresponding rows and columns of a determinant are interchanged, the value of the determinant remains unchanged.*

PROOF. Let the given determinant of order n be represented by Δ_n as in (2) above. Interchanging the corresponding rows and columns of Δ_n, we obtain the determinant

$$\Delta_n' = \begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ c_1 & c_2 & \cdots & c_n \\ \cdot & \cdot & \cdot & \cdot \\ l_1 & l_2 & \cdots & l_n \end{vmatrix}$$

whose leading term $a_1b_2c_3 \cdots l_n$ is the same as the leading term of Δ_n. In Δ_n', the letters denote the rows and the subscripts the columns, the converse of their roles in Δ_n. Hence, by keeping the subscripts of $a_1b_2c_3 \cdots l_n$ in their normal order and permuting the n letters in $n!$ different ways, we obtain every one of the terms in the expansion of Δ_n. Furthermore, identical terms in both determinants have the same signs by considering inversions among the letters and not the subscripts in Δ_n'. Hence $\Delta_n' = \Delta_n$, and the theorem is established.

As an immediate consequence of this theorem we have the important

Corollary. *Any theorem on determinants which is true for its rows is also valid for its columns.*

NOTE 3. In operating with determinants, the student will note a definite pattern of symmetry between rows and columns.

Theorem 4. *If every element of any row (or column) is zero, the value of the determinant is zero.*

PROOF. The theorem follows immediately from the expansion of the determinant. For, each term in the expansion of Δ_n must contain a factor which is an element from the row of zeros. Hence each term is zero and $\Delta_n = 0$.

Theorem 5. *If two rows (or columns) of a determinant are interchanged, the value of the determinant is changed in sign but not numerically.*

PROOF. The interchange of two rows results in the interchange of two subscripts in each term of the expansion of the determinant. But by Theorem 1, the sign of each term is then changed. Hence the determinant is changed in sign without altering its numerical value.

Theorem 6. *If the corresponding elements of two rows (or columns) of a determinant are equal, the value of the determinant is zero.*

PROOF. Let Δ_n have two identical rows. If these two rows are interchanged, Δ_n changes in value to $-\Delta_n$ by Theorem 5. But the interchange of two *identical* rows leaves the determinant unaltered. Hence, $\Delta_n = -\Delta_n$ from which $2\Delta_n = 0$ and $\Delta_n = 0$.

Theorem 7. *If each element of any row (or column) of a determinant is multiplied by the same number k, the new determinant has a value equal to k times that of the original determinant.*

PROOF. Let the original and new determinants be denoted by Δ_n and Δ_n', respectively. Since each term in the expansion of a determinant contains one element and only one element from each row, each term in the expansion of Δ_n' is k times the corresponding term of Δ_n. Hence $\Delta_n' = k\Delta_n$.

Corollary. *If all the elements of any row (or column) have a common factor k, then k is a factor of the determinant. This common factor k may then be removed from each element and placed as a multiplier in front of the resulting determinant.*

Theorem 8. *If each element of some row (or column) of a determinant is the sum of two quantities, the determinant may be written as the sum of two determinants, that is,*

$$\begin{vmatrix} a_1 + a_1' & b_1 & \cdots & l_1 \\ a_2 + a_2' & b_2 & \cdots & l_2 \\ \cdots & \cdots & \cdots & \cdots \\ a_n + a_n' & b_n & \cdots & l_n \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & \cdots & l_1 \\ a_2 & b_2 & \cdots & l_2 \\ \cdots & \cdots & \cdots & \cdots \\ a_n & b_n & \cdots & l_n \end{vmatrix} + \begin{vmatrix} a_1' & b_1 & \cdots & l_1 \\ a_2' & b_2 & \cdots & l_2 \\ \cdots & \cdots & \cdots & \cdots \\ a_n' & b_n & \cdots & l_n \end{vmatrix}.$$

PROOF. Let the three determinants, taken in order, be designated by Δ, Δ_n, and Δ_n', respectively. Then we are to show that

$$\Delta = \Delta_n + \Delta_n'.$$

In the relation to be established, we have indicated the first column as having each of its elements the sum of two quantities. The proof for any other column (or row) proceeds in exactly the same way.

Now in accordance with the definition of a determinant, the expansion of Δ may be written in the form

$$\begin{aligned} \Delta &= (a_1 + a_1')A_1 + (a_2 + a_2')A_2 + \cdots + (a_n + a_n')A_n \\ &= (a_1A_1 + a_2A_2 + \cdots + a_nA_n) + (a_1'A_1 + a_2'A_2 + \cdots + a_n'A_n), \end{aligned}$$

where $A_1, A_2, \cdots, A_n$ are expressions containing no elements from the first column.

But, by the definition of a determinant and the significance of $A_1, A_2, \cdots, A_n$, it follows that

$$\Delta_n = a_1A_1 + a_2A_2 + \cdots + a_nA_n$$

and

$$\Delta_n' = a_1'A_1 + a_2'A_2 + \cdots + a_n'A_n,$$

whence $\Delta = \Delta_n + \Delta_n'$, as was to be shown.

Corollary. *If each element of some row (or column) of a determinant is the sum of three (or more) quantities, this determinant may be written as the sum of three (or more) determinants.*

We now establish a theorem which is very useful in evaluating determinants.

Theorem 9. *If each element of any row (or column) of a determinant is multiplied by the same number k and added to the corresponding element of another row (or column), the value of the determinant remains unchanged.*

PROOF. For convenience, and to fix our ideas, we will establish a specific example of the stated theorem. The proof for any other column (or row) is precisely the same. Hence, we will show that

$$\begin{vmatrix} a_1 + kb_1 & b_1 & c_1 & \cdots & l_1 \\ a_2 + kb_2 & b_2 & c_2 & \cdots & l_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n + kb_n & b_n & c_n & \cdots & l_n \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 & \cdots & l_1 \\ a_2 & b_2 & c_2 & \cdots & l_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n & b_n & c_n & \cdots & l_n \end{vmatrix}.$$

By Theorem 8,

$$\begin{vmatrix} a_1 + kb_1 & b_1 & \cdots & l_1 \\ a_2 + kb_2 & b_2 & \cdots & l_2 \\ \cdots & \cdots & \cdots & \cdots \\ a_n + kb_n & b_n & \cdots & l_n \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & \cdots & l_1 \\ a_2 & b_2 & \cdots & l_2 \\ \cdots & \cdots & \cdots & \cdots \\ a_n & b_n & \cdots & l_n \end{vmatrix} + \begin{vmatrix} kb_1 & b_1 & \cdots & l_1 \\ kb_2 & b_2 & \cdots & l_2 \\ \cdots & \cdots & \cdots & \cdots \\ kb_n & b_n & \cdots & l_n \end{vmatrix}$$

By Theorem 7, Corollary,

$$= \begin{vmatrix} a_1 & b_1 & \cdots & l_1 \\ a_2 & b_2 & \cdots & l_2 \\ \cdots & \cdots & \cdots & \cdots \\ a_n & b_n & \cdots & l_n \end{vmatrix} + k \begin{vmatrix} b_1 & b_1 & \cdots & l_1 \\ b_2 & b_2 & \cdots & l_2 \\ \cdots & \cdots & \cdots & \cdots \\ b_n & b_n & \cdots & l_n \end{vmatrix}$$

By Theorem 6,

$$= \begin{vmatrix} a_1 & b_1 & \cdots & l_1 \\ a_2 & b_2 & \cdots & l_2 \\ \cdots & \cdots & \cdots & \cdots \\ a_n & b_n & \cdots & l_n \end{vmatrix}.$$

We will now establish an important theorem which was stated without proof in Sec. 15.4 and was then used for evaluating determinants. Before studying the proof of this theorem, the student should reread the definitions of the terms *minor* and *cofactor* and the verification of this theorem for Δ_3, as given in Sec. 15.4.

Theorem 10. *The value of any determinant of order n is equal to the sum of n products each of which is formed by multiplying each element of any row (or column) by its corresponding cofactor.*

PROOF. We will establish the theorem by considering the expansion of the determinant

$$\Delta_n = \begin{vmatrix} a_1 & b_1 & c_1 & \cdots & l_1 \\ a_2 & b_2 & c_2 & \cdots & l_2 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_n & b_n & c_n & \cdots & l_n \end{vmatrix}$$

with respect to the elements of the first row. The proof is the same for any other row (or column).

Accordingly, we are to show that

$$\Delta_n = a_1A_1 + b_1B_1 + c_1C_1 + \cdots + l_1L_1, \tag{3}$$

where $A_1, B_1, C_1, \cdots, L_1$ are the respective cofactors of the elements $a_1, b_1, c_1, \cdots, l_1$.

The proof consists of two steps in which it is shown that (1) the terms in the expansion (3) include all the $n!$ products given by the definition of Δ_n and that (2) each of these products has the proper sign.

(1) The cofactor A_1 is a determinant of order $n - 1$ and the terms of its expansion consist of $(n - 1)!$ products, each of which contains no element from the first row and first column. Hence a_1A_1 consists of $(n - 1)!$ products, each of which contains one element and only one from each column and from each row, including the first row and first column. Similarly, b_1B_1 consists of $(n - 1)!$ products, each of which contains one element and only one from each column and from each row, including the first row and second column. Continuing this way, we see that for the n terms of (3) we have a total of $n(n - 1)! = n!$ products, each of which contains one element and only one element from each column and from each row of Δ_n. This is in accordance with the general definition of a determinant.

(2) The signs of the terms of the cofactor A_1 are in accordance with the definition of A_1 and are the same for the expansion of a_1A_1, for the factor a_1 does not change the number of inversions. Note also that for a_1, the element in the first row and first column, the sum of the number of the row and the number of the column is $1 + 1 = 2$, an even number.

In general, consider an element of Δ_n in the ith row and jth column. This element can be brought into the position occupied by the element a_1 by $i - 1$ successive interchanges of adjacent rows and $j - 1$ successive interchanges of adjacent columns, a total number of $i + j - 2$ successive interchanges. By Theorem 1, each such interchange changes the sign of a term once. Hence if $i + j - 2$ is even, $i + j$ is even, and the term is preceded by a plus sign; if $i + j - 2$ is odd, $i + j$ is odd, and the term is preceded by a minus sign.

This completes the proof.

Corollary. *If, in the expansion of a determinant with respect to the elements of any row (or column), the elements of this row (or column) are replaced by the corresponding elements of any other row (or column), the resulting expression is equal to zero.*

This follows from the fact that the resulting expression is then the expansion of a determinant with two identical rows (or columns) and hence, by Theorem 6, is equal to zero.

Thus, in the expansion of Δ_n as given by (3) above, if we replace the elements of the first row by the elements of the second row, we have

$$a_2A_1 + b_2B_1 + c_2C_1 + \cdots + l_2L_1 = \begin{vmatrix} a_2 & b_2 & c_2 & \cdots & l_2 \\ a_2 & b_2 & c_2 & \cdots & l_2 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_n & b_n & c_n & \cdots & l_n \end{vmatrix} = 0.$$

NOTE 4. In connection with the proof in step (2) of Theorem 10, the student should note Ex. 16 of Group 55, Sec. 15.4.

With the completion of the proofs of Theorems 9 and 10, we have justified the procedure for evaluating any determinant, as given in Sec. 15.4.

EXERCISES. GROUP 56

1. Show that the expansion of a determinant of order 2, as given by relation (1) of Sec. 15.3, is in accordance with the general definition of Sec. 15.5 for a determinant of any order.

2. Show that the expansion of a determinant of order 3, as given by relation (2) of Sec. 15.4, is in accordance with the general definition of Sec. 15.5 for a determinant of any order.

3. Verify Theorem 2 (Sec. 15.5) for determinants of orders 2 and 3.

4. Establish the corollary to Theorem 7 (Sec. 15.5).

5. Establish the corollary to Theorem 8 (Sec. 15.5).

6. If the corresponding elements of two rows (or columns) of a determinant are proportional, show that the value of the determinant is zero.

7. Establish Theorem 4 by means of Theorems 9 and 6 (Sec. 15.5).

8. Show that

$$\begin{vmatrix} a_1 & 0 & 0 & 0 \\ a_2 & b_2 & 0 & 0 \\ a_3 & b_3 & c_3 & 0 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1b_2c_3d_4.$$

In each of Exs. 9–17, evaluate the given determinant.

9. $\begin{vmatrix} 5 & 10 & 1 \\ \bar{1} & \bar{1} & \bar{3} \\ 4 & 7 & 5 \end{vmatrix}.$ 10. $\begin{vmatrix} 3 & 2 & \bar{4} \\ 2 & \bar{1} & 7 \\ 6 & 4 & \bar{8} \end{vmatrix}.$ 11. $\begin{vmatrix} 1 & x & \bar{y} \\ \bar{x} & 1 & z \\ y & \bar{z} & 1 \end{vmatrix}.$

12. $\begin{vmatrix} 2 & \bar{3} & 2 & 1 \\ 8 & 2 & \bar{5} & 4 \\ \bar{4} & 0 & 7 & \bar{2} \\ 6 & \bar{9} & 6 & 3 \end{vmatrix}.$ 13. $\begin{vmatrix} 1 & 0 & 0 & 0 \\ \bar{2} & 2 & 0 & 0 \\ 5 & 9 & \bar{3} & 0 \\ 7 & \bar{6} & 2 & 4 \end{vmatrix}.$

14. $\begin{vmatrix} 2 & 3 & 1 & 3 \\ \bar{1} & 5 & 2 & 7 \\ 1 & 3 & 1 & 3 \\ \bar{1} & 10 & 4 & 13 \end{vmatrix}.$ 15. $\begin{vmatrix} 4 & 4 & 5 & 2 \\ 0 & 2 & 1 & 1 \\ 4 & 3 & 6 & 2 \\ 1 & 1 & 2 & 1 \end{vmatrix}.$

16. $\begin{vmatrix} 3 & 5 & \bar{1} & 4 & 1 \\ 6 & 9 & \bar{2} & 7 & 2 \\ 8 & 14 & \bar{2} & 11 & 2 \\ 3 & 5 & \bar{2} & 5 & 1 \\ 6 & 9 & 2 & 3 & \bar{1} \end{vmatrix}.$ 17. $\begin{vmatrix} 1 & 4 & \bar{3} & 2 & 5 \\ 3 & 0 & 2 & 1 & \bar{1} \\ 2 & 4 & \bar{2} & 0 & 6 \\ 3 & 1 & 7 & 1 & 5 \\ 2 & 3 & 4 & 1 & 0 \end{vmatrix}.$

In each of Exs. 18–19, verify the given relation without actually expanding the determinants.

18. $$\begin{vmatrix} 3 & 2 & 9 & 3 \\ 9 & 3 & \bar{7} & 5 \\ 7 & 1 & 3 & 2 \\ \bar{5} & 0 & 6 & 1 \end{vmatrix} + \begin{vmatrix} 3 & 2 & \bar{4} & 3 \\ 9 & 3 & 5 & 5 \\ 7 & 1 & 2 & 2 \\ \bar{5} & 0 & \bar{3} & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 & 5 & 3 \\ 9 & 3 & \bar{2} & 5 \\ 7 & 1 & 5 & 2 \\ \bar{5} & 0 & 3 & 1 \end{vmatrix}.$$

19. $$\begin{vmatrix} 2 & \bar{1} & 1 & \bar{1} \\ 3 & 5 & 3 & 10 \\ 1 & 2 & 1 & 4 \\ 3 & 7 & 3 & 13 \end{vmatrix} + \begin{vmatrix} 2 & 1 & \bar{1} & \bar{1} \\ 3 & 3 & 5 & 10 \\ 1 & 1 & 2 & 4 \\ 3 & 3 & 7 & 13 \end{vmatrix} = 0.$$

20. Show that $x + y + z$ is a factor of the determinant

$$\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}.$$

21. It is shown in analytic geometry that the equation of the circle passing through the three given noncollinear points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, and $P(x_3, y_3)$ may be written in the form

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0.$$

Verify the fact that the coordinates of each of the points P_1, P_2, and P_3 satisfy this equation.

22. By means of Ex. 21, find the equation of the circle passing through the three points (0, 0), (3, 6), (7, 0).

23. By means of Ex. 21, find the equation of the circle passing through the three points (2, −2), (−1, 4), (4, 6).

24. By means of Ex. 21 show that the four points (−1, −1), (2, 8), (5, 7), (7, 3) lie on a circle. Such points are said to be *concyclic.*

25. It is shown in solid analytic geometry that the equation of the plane passing through the three given noncollinear points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, and $P_3(x_3, y_3, z_3)$ may be written in the form

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

Verify the fact that the coordinates of each of the points P_1, P_2, and P_3 satisfy this equation.

26. By means of Ex. 25, find the equation of the plane passing through the three points (6, 2, 0), (4, −1, 2), and (3, 4, −1).

27. If no three of the four points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4) are collinear, show by means of Ex. 25 that if these points are coplanar then

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0.$$

28. By means of Ex. 27, show that the four points (1, 0, −4), (2, −1, 3), (−2, 3, 5), and (−1, 2, 4) are coplanar.

29. It is shown in solid analytic geometry that the volume V of a tetrahedron whose vertices are $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$, and $P_4(x_4, y_4, z_4)$ is given by the formula

$$V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

where the absolute value of the determinant is to be taken. Use this result to find the volume of a tetrahedron whose vertices are (−4, 6, 3), (8, −3, 5), (4, 0, −1), and (5, 3, 9).

30. Show that if the elements of a determinant Δ are polynomials in x, and if $\Delta = 0$ when $x = r$, then $x - r$ is a factor of the expansion of Δ.

In each of Exs. 31–33, factor the given determinant.

31. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$.

32. $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$.

33. $\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}$.

34. Show that $\begin{vmatrix} 1 & x & x^2 - yz \\ 1 & y & y^2 - xz \\ 1 & z & z^2 - xy \end{vmatrix} = 0.$

35. Show that $\begin{vmatrix} y^2 + z^2 & xy & xz \\ xy & x^2 + z^2 & yz \\ xz & yz & x^2 + y^2 \end{vmatrix} = 4x^2y^2z^2.$

15.6. SYSTEMS OF LINEAR EQUATIONS

At this point the student will find it helpful to reread Sec. 4.7 wherein we discussed a system of two or more linear equations in the same number of variables but without reference to determinants. In this section we shall study the solution and some of the properties of various systems of linear equations from the standpoint of determinants. We will start with the general case of Cramer's rule, which was previously discussed for systems of 2 and 3 equations in Secs. 15.3 and 15.4, respectively.

Consider, therefore, the following system of n linear equations in n variables:

$$(1)\quad \begin{aligned} a_1x + b_1y + c_1z + \cdots + l_1w &= k_1, \\ a_2x + b_2y + c_2z + \cdots + l_2w &= k_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_nx + b_ny + c_nz + \cdots + l_nw &= k_n. \end{aligned}$$

In order to form the proper determinants from the coefficients, it is necessary to write the equations of a system as indicated in (1), that is, with the same variables under each other and the constant terms on the right-hand side. The equations are then said to be *in order*. If any variable is missing, its position is left vacant and its coefficient is taken as zero.

The determinant whose elements are the coefficients in (1) is called the *determinant of the system* and is designated by Δ, that is,

$$(2)\quad \Delta = \begin{vmatrix} a_1 & b_1 & \cdots & l_1 \\ a_2 & b_2 & \cdots & l_2 \\ \cdots & \cdots & \cdots & \cdots \\ a_n & b_n & \cdots & l_n \end{vmatrix}.$$

To determine the value of any variable in (1), we must eliminate all the other variables. This may be done very conveniently by means of cofactors. Let $A_1, A_2, \cdots, A_n$ be the respective cofactors of $a_1, a_2, \cdots, a_n$, the elements of the first column of Δ. Multiplying both sides of each of the equations of the system (1) by $A_1, A_2, \cdots, A_n$, respectively, we obtain

$$(3)\quad \begin{aligned} a_1A_1x + b_1A_1y + c_1A_1z + \cdots + l_1A_1w &= k_1A_1, \\ a_2A_2x + b_2A_2y + c_2A_2z + \cdots + l_2A_2w &= k_2A_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_nA_nx + b_nA_ny + c_nA_nz + \cdots + l_nA_nw &= k_nA_n. \end{aligned}$$

Adding the equations of the system (3), member by member, we obtain

$$(4)\quad (a_1A_1 + a_2A_2 + \cdots + a_nA_n)x + (b_1A_1 + b_2A_2 + \cdots + b_nA_n)y + \cdots + (l_1A_1 + l_2A_2 + \cdots + l_nA_n)w = k_1A_1 + k_2A_2 + \cdots + k_nA_n.$$

The coefficient of x in (4) is the expansion of Δ with respect to the elements of the first column (Theorem 10, Sec. 15.5). The coefficient of y is the coefficient of x with the elements of the first column replaced by the corresponding elements of the second column. Hence, by the corollary of Theorem 10, Sec. 15.5, the coefficient of y is equal to zero. Similarly, the coefficients of the remaining variables $z, \cdots, w$ in (4) are equal to zero. The right member of (4) is the expansion of Δ with the elements of its first column replaced by the corresponding right members of the system (1). We will designate this last expansion by Δ_1, that is,

$$\Delta_1 = \begin{vmatrix} k_1 & b_1 & \cdots & l_1 \\ k_2 & b_2 & \cdots & l_2 \\ \cdots & \cdots & \cdots & \cdots \\ k_n & b_n & \cdots & l_n \end{vmatrix}.$$

Hence equation (4) may be written

$$\Delta x = \Delta_1$$

whence

$$x = \frac{\Delta_1}{\Delta}, \quad \text{provided that } \Delta \neq 0.$$

Similarly, we may solve for the remaining variables. Thus, let B_1, $B_2, \cdots, B_n$ be the respective cofactors of $b_1, b_2, \cdots, b_n$, the elements in the second column of Δ. If we multiply both sides of each of the equations of the system (1) by $B_1, B_2, \cdots, B_n$, respectively, add the resulting equations member by member, and apply Theorem 10 of Sec. 15.5 and its Corollary, we obtain

$$\Delta y = \Delta_2,$$

where Δ_2 is the determinant obtained from Δ by replacing the elements of its second column by the corresponding right members of the system (1).

Hence,

$$y = \frac{\Delta_2}{\Delta}, \quad \text{provided that } \Delta \neq 0.$$

Similarly,

$$z = \frac{\Delta_3}{\Delta}, \cdots, w = \frac{\Delta_n}{\Delta}, \Delta \neq 0.$$

Conversely, we may show by actual substitution that this solution satisfies each of the equations of the system (1).

We state these results as

Theorem 11. (*Cramer's Rule*). *For the system of n linear equations in n variables,*

$$a_1x + b_1y + c_1z + \cdots + l_1w = k_1,$$
$$a_2x + b_2y + c_2z + \cdots + l_2w = k_2,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_nx + b_ny + c_nz + \cdots + l_nw = k_n,$$

let Δ be the determinant of the system and let Δ_i, $i = 1, 2, \cdots, n$, be the determinant obtained from Δ by replacing the elements of its ith column by the corresponding right members of the system.

Then, if $\Delta \neq 0$, the system has the unique solution

$$x = \frac{\Delta_1}{\Delta}, \quad y = \frac{\Delta_2}{\Delta}, \quad \cdots, \quad w = \frac{\Delta_n}{\Delta}.$$

As an illustration of Theorem 11, we have

Example 1. By Cramer's rule, solve the system

$$\begin{aligned} 3x + 2y + z - 2w &= 4, \\ 2x - y + 2z - 5w &= 15, \\ 4x + 2y - w &= 1, \\ 3x - 2z - 4w &= 1. \end{aligned}$$

SOLUTION. The first step is to verify the fact that the given system is in order.

The next step is to evaluate the determinant of the system. We find

$$\Delta = \begin{vmatrix} 3 & 2 & 1 & \bar{2} \\ 2 & \bar{1} & 2 & \bar{5} \\ 4 & 2 & 0 & \bar{1} \\ 3 & 0 & \bar{2} & \bar{4} \end{vmatrix} = -65.$$

Since $\Delta \neq 0$, the given system has a unique solution which, by Cramer's rule, is

$$x = \frac{\Delta_1}{\Delta} = \frac{\begin{vmatrix} 4 & 2 & 1 & \bar{2} \\ 15 & \bar{1} & 2 & \bar{5} \\ 1 & 2 & 0 & \bar{1} \\ 1 & 0 & \bar{2} & \bar{4} \end{vmatrix}}{-65} = \frac{-65}{-65} = 1,$$

$$y = \frac{\Delta_2}{\Delta} = \frac{\begin{vmatrix} 3 & 4 & 1 & \bar{2} \\ 2 & 15 & 2 & \bar{5} \\ 4 & 1 & 0 & \bar{1} \\ 3 & 1 & \bar{2} & \bar{4} \end{vmatrix}}{-65} = \frac{130}{-65} = -2,$$

$$z = \frac{\Delta_3}{\Delta} = \frac{\begin{vmatrix} 3 & 2 & 4 & \bar{2} \\ 2 & \bar{1} & 15 & \bar{5} \\ 4 & 2 & 1 & \bar{1} \\ 3 & 0 & 1 & \bar{4} \end{vmatrix}}{-65} = \frac{-195}{-65} = 3,$$

$$w = \frac{\Delta_4}{\Delta} = \frac{\begin{vmatrix} 3 & 2 & 1 & 4 \\ 2 & \bar{1} & 2 & 15 \\ 4 & 2 & 0 & 1 \\ 3 & 0 & \bar{2} & 1 \end{vmatrix}}{-65} = \frac{65}{-65} = -1.$$

It is easily verified that this solution satisfies all the equations of the given system.

It is evident from Cramer's rule that if the determinant of the system $\Delta = 0$, there cannot be a unique solution

(5) $$x = \frac{\Delta_1}{\Delta}, \quad y = \frac{\Delta_2}{\Delta}, \cdots, \quad w = \frac{\Delta_n}{\Delta},$$

for division by zero is an excluded operation. Furthermore, if we write the solution (5) in the form

$$\Delta x = \Delta_1, \Delta y = \Delta_2, \cdots, \Delta w = \Delta_n,$$

it follows that if $\Delta = 0$, then $\Delta_i = 0$, $i = 1, 2, \cdots, n$. Hence, if at least one of the determinants Δ_i is different from zero, we have a contradiction, and there is no solution; the system is then said to be *inconsistent*. If, however, the determinants $\Delta_1, \Delta_2, \cdots, \Delta_n$ are all equal to zero, it can be shown that there may be infinitely many solutions; the system is then said to be *dependent*. We have already discussed inconsistent and dependent systems for two equations in two variables in Sec. 4.7. But the complete discussion of the general system of n linear equations in n variables when $\Delta = 0$ is beyond the scope of this book. However, for convenient reference, we record the following facts:

For a system of n linear equations in n variables, let Δ be the determinant of the system and let Δ_i, $i = 1, 2, \cdots, n$, be the determinant obtained from Δ by replacing the elements of the ith column by the corresponding right members of the system.

1. *If $\Delta \neq 0$, the system has a unique solution given by Cramer's rule. The system is then said to be consistent.*

2. *If* $\Delta = 0$ *and* $\Delta_i \neq 0$ *for at least one i, the system has no solution and is said to be inconsistent.*

3. *If* $\Delta = 0$ *and* $\Delta_i = 0$ *for all values of i, the system has either no solution and is inconsistent or else infinitely many solutions and is said to be dependent.*

In the linear system (1) if at least one constant term is different from zero, the system is said to be *nonhomogeneous.* But if all the constant terms (the k's) are zero, the system is said to be *homogeneous* and takes the form

$$
\begin{aligned}
a_1x + b_1y + \cdots + l_1w &= 0,\\
a_2x + b_2y + \cdots + l_2w &= 0,\\
\cdots\cdots\cdots&\cdots\\
a_nx + b_ny + \cdots + l_nw &= 0.
\end{aligned}
\tag{6}
$$

It is clear that the system (6) is satisfied if each variable has the value zero, and this is true whether the determinant Δ of the system is zero or not. Since a homogeneous system always has a zero solution, it is called the *trivial solution.* If $\Delta \neq 0$, a homogeneous system has the trivial solution as its only solution by Cramer's rule. Hence, if a homogeneous system is to have solutions other than the trivial solution, Δ cannot be different from zero. As a matter of fact, the following theorem is established in advanced treatises.

Theorem 12. *A homogeneous linear system of n equations in n variables has solutions other than the trivial solution of zeros if and only if the determinant of the system is equal to zero.*

Example 2. Solve the homogeneous system

$$
\begin{aligned}
2x + 3y - z &= 0,\\
x - y - 3z &= 0,\\
x + 3y + z &= 0.
\end{aligned}
$$

SOLUTION. The determinant of the given system is readily found to be zero so that there are solutions other than the trivial solution. To obtain such solutions we proceed as follows:

We attempt, if possible, to solve two of the equations for two of the variables in terms of the third variable. Thus, rewriting the first two equations in the form

$$
\begin{aligned}
2x + 3y &= z,\\
x - y &= 3z,
\end{aligned}
$$

we find that we may solve for x and y in terms of z, for the determinant of this system

$$\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = -5 \neq 0.$$

We thus obtain $x = 2z$, $y = -z$. These values of x and y satisfy the third equation identically, for $2z - 3z + z = 0$ for all values of z.

Hence we may obtain as many solutions as we please by assigning arbitrary values to z and computing the corresponding values of x and y. Thus,

$$\text{For } z = 1,\ x = 2z = 2 \text{ and } y = -z = -1.$$
$$\text{For } z = 2,\ x = 4 \text{ and } y = -2, \text{ and so on.}$$

Evidently all nonzero solutions for x, y, and z, respectively, are in the ratio $2:-1:1$.

Example 3. Solve the homogeneous system

$$\begin{aligned} x - y + 2z &= 0, \\ 2x - 2y + 4z &= 0, \\ 3x - 3y + 6z &= 0. \end{aligned}$$

SOLUTION. The determinant of the given system is zero. If we now attempt to obtain nontrivial solutions as in the previous example, we encounter a difficulty because the minors of all the elements of the determinant of the system are also zero. We observe, however, that the three equations are equivalent, for the second and third equations may be obtained from the first by multiplying, respectively, by 2 and 3. Hence, if we solve the first equation for x in terms of y and z, we find

$$x = y - 2z,$$

which we may use as a formula for obtaining values of x corresponding to arbitrarily chosen values for y and z. Thus,

$$\text{For } y = 1 \text{ and } z = 1,\ x = -1.$$
$$\text{For } y = 2 \text{ and } z = 1,\ x = 0, \text{ and so on.}$$

Up to this point, the linear systems discussed had as many equations as variables. If the number of equations and the number of variables differ, the theory becomes involved and the complete treatment requires advanced study. However, there are several situations we can discuss.

First, we will consider a system where the number of equations is less than the number of variables; such a system is said to be *defective*. As a rule, a defective system has infinitely many solutions. The simplest example of such a system is a single equation in two variables. Thus, $x + 2y = 6$ has infinitely many solutions obtained by assigning arbitrary values to either variable and then computing the corresponding values of the other variable.

In general, for a defective system of n equations in m variables, where $n < m$, we may be able to solve for n of these variables in terms of the remaining $m - n$ variables. Then by assigning arbitrary values to these $m - n$ variables, we can obtain the corresponding values of the other n variables. This process is illustrated in

Example 4. Find solutions of the defective system

$$x - 2y + z = 1,$$
$$x + y + 4z = 1.$$

SOLUTION. Here it is possible to solve for x and y in terms of z. We find

$$x = 1 - 3z, \qquad y = -z.$$

Hence, by assigning arbitrary values to z, we may obtain the corresponding values of x and y and obtain as many solutions as we desire. Thus,

For $\quad z = 0,\ x = 1,\quad y = 0,$

$\quad z = 1,\ x = -2,\ y = -1,$ and so on.

We next consider a system in which the number of equations is more than the number of variables; such a system is said to be *redundant*. Say that we have a system of n equations in m variables, where $n > m$. It may be possible to solve m of these equations for the m variables. If this solution satisfies *all* of the remaining $n - m$ equations, the given system is consistent, otherwise it is inconsistent.

A redundant system of particular interest is one in which the number of equations exceeds the number of variables by one. We will illustrate this by the following system of three equations in two variables:

$$a_1x + b_1y = k_1,$$
$$a_2x + b_2y = k_2,$$
$$a_3x + b_3y = k_3.$$

We wish to determine under what conditions this system is consistent, that is, has a common solution. The solution of the first two equations by Cramer's rule is

$$x = \frac{\begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}}{\Delta}, \qquad y = \frac{\begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}}{\Delta}, \qquad \Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0.$$

This solution must satisfy the third equation, that is, we must have

$$a_3 \frac{\begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} + b_3 \frac{\begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} - k_3 = 0.$$

Clearing of fractions, we have

$$a_3 \begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix} + b_3 \begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix} - k_3 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0.$$

Changing the signs of all the terms, we may write

$$a_3 \begin{vmatrix} b_1 & k_1 \\ b_2 & k_2 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix} + k_3 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0.$$

The left member is now seen to be the expansion of the following determinant with respect to the elements of the third row (Theorem 10, Sec. 15.5):

$$\Delta_3 = \begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}.$$

This determinant Δ_3 is called the *eliminant* of the system.

Hence, a necessary condition for the given system to be consistent is that $\Delta_3 = 0$. This result may be extended to n equations in $n - 1$ variables as stated in

Theorem 13. *A necessary condition that a redundant nonhomogeneous linear system of n equations in n − 1 variables be consistent is that the determinant of order n formed from the coefficients and the constant terms have the value zero.*

NOTE. The converse of Theorem 13 is not necessarily true; hence the condition is not sufficient. For example, in the system

$$x + 2y = 5,$$
$$2x + 4y = 9,$$
$$3x + 6y = 12.$$

the eliminant is zero, but the system is not consistent. In fact, no two of the three equations are consistent.

Example 5. Find the value of k for which the following redundant system is consistent, and find the solution of the system:

$$\begin{aligned} 2x + y + z &= k, \\ x - y - 2z &= -2, \\ 3x - y + z &= 2k, \\ x + 2y + z &= 1. \end{aligned}$$

SOLUTION. For this system to be consistent, we must have, by Theorem 13,

$$\begin{vmatrix} 2 & 1 & 1 & k \\ 1 & \bar{1} & \bar{2} & \bar{2} \\ 3 & \bar{1} & 1 & 2k \\ 1 & 2 & 1 & 1 \end{vmatrix} = 0.$$

The expansion of this determinant gives us a value of 3 for k. Setting k equal to 3 in the given system and solving the first three equations, we find $x = 1$, $y = -1$, $z = 2$. This solution is readily found to satisfy the fourth equation.

EXERCISES. GROUP 57

1. Verify the values shown for all the determinants of Example 1 (Sec. 15.6) and check the solution.

In each of Exs. 2–9, solve the given system by Cramer's rule and check the solution by substitution.

2. $x + 3y - z = 0, \quad 3x - 4y + z = 2, \quad 2x + 2y + z = 13.$
3. $2x + 2y - z = 2, \quad x - 3y - 2z = 2, \quad 3x + 4y + z = 7.$
4. $3x - 4y + 7z = 4, \quad x + 2y - 5z = 8, \quad 2x - 3y + 9z = 2.$
5. $x + 5y + 4z = 1, \quad 2x - 5y + 3z = -3, \quad x + 9y + 5z = 2.$
6. $$\begin{aligned} 4x + 2y + 3z + w &= 3, \\ 2x - 3y - w &= 2, \\ 3x - 2y + z + 2w &= 0, \\ x + 3z - 5w &= 1. \end{aligned}$$
7. $$\begin{aligned} x + 2y + z - 2w &= -2, \\ 3x - y - z + w &= 3, \\ 2x - y + 2z - 4w &= 1, \\ 4x - 3y - 2z + w &= 3. \end{aligned}$$
8. $$\begin{aligned} x + 3y + 2z + u - v &= 1, \\ 2x - 5y - z - u + 2v &= 5, \\ x + 7y + z - 2v &= 1, \\ 3x - 3y + 2u + 4v &= 1, \\ x + 4y - z - 2u &= 5. \end{aligned}$$
9. $$\begin{aligned} x + 4y - 3z + 2u - 3v &= 2, \\ 2x - 5z - 3u + 2v &= -2, \\ 3x + 2y + 7z + u &= 6, \\ x - 3y - 2u + 3v &= 1, \\ 2x - 5y + 3z - v &= 7. \end{aligned}$$

In each of Exs. 10 and 11, show that the given system has no unique solution.

10. $x + y + z + 7w = 4,$
$3x + 8y - 2z + w = -1,$
$3x + 7y - z + 5w = 11,$
$x + 3y - z + w = 3.$

11. $3x + y - z + 4w = 5,$
$x + y + 3z + 5w = 8,$
$x - 5y - 11z = -2,$
$x + 3y + 5z + 2w = 9.$

12. If a homogeneous linear system of n equations in n variables has a solution $x = \alpha_1,\ y = \alpha_2, \cdots, w = \alpha_n$, show that it also has the solution $x = k\alpha_1$, $y = k\alpha_2, \cdots, w = k\alpha_n$, where k is an arbitrary constant.

In each of Exs. 13 and 14, show that the given system has only a trivial solution of zeros.

13. $x + 3y + 2z + w = 0,$
$2x - y + 4z + 3w = 0,$
$3x + 7y + 6z + 4w = 0,$
$2x + 3y + 7z + 5w = 0.$

14. $2x + 4y - z + 3w = 0,$
$x + 6y + 2z - 5w = 0,$
$3x - 4z + 3w = 0,$
$4x - 2y + 3z + w = 0.$

In each of Exs. 15 and 16, show that the given system has solutions other than the trivial solution, and find several such solutions.

15. $2x + 2y + 3z - w = 0,$
$x - y + 2z + w = 0,$
$3x + 2y + z - 2w = 0,$
$x + y - 3z - 2w = 0.$

16. $x - 2y + 2z - w = 0,$
$3x + 2y + 4z + 2w = 0,$
$x + 3y + z + 2w = 0,$
$2x - y + z + w = 0.$

In each of Exs. 17 and 18, solve the given defective system for x, y, and z in terms of w and obtain several solutions.

17. $x + y + z + w = 3,$
$x - 2y + 3z + 2w = -4,$
$2x - y - 2z - 2w = 0.$

18. $2x + 3y - z + w = 2,$
$x + y - z + 2w = 4,$
$3x - 2y - 4z + w = 6.$

19. A group of 18 men, women, and boys earn \$25 per hour. The men earn \$2 per hour, the women \$1.50 per hour, and the boys \$1 per hour. Find the number of men, women, and boys.

In each of Exs. 20 and 21, determine whether the given redundant system is consistent or inconsistent. If consistent, find the solution.

20. $2x + 2y - z = -5,$
$x - y + 3z = 6,$
$2x - 4y + 3z = 1,$
$x + y + z = 4.$

21. $2x + y - z = 7,$
$x - y - z = 0,$
$x + 2y + z = 8,$
$3x - 2y - 2z = 3.$

22. Check all the details of Example 5 of Sec. 15.6.

In each of Exs. 23 and 24, find the value of k for which the given redundant system is consistent, and find the solution of the system.

23. $2x + y + 3z = 3,$
$x - y - 2z = 2k,$
$x + 2y + 2z = 4k,$
$x + y + z = 3.$

24. $x + y - 3z = k,$
$3x + 3y + z = 4,$
$2x - y - 4z = 4,$
$x - y - 3z = -k.$

25. By actual substitution, show that the solution by Cramer's rule (Theorem 11) satisfies the first equation of the system (1) in Sec. 15.6.

16

Logarithms

16.1. INTRODUCTION

In this chapter we consider some of the properties and uses of the logarithmic function. Since this is a text book on algebra, the student may naturally inquire why we are including a nonalgebraic function (Sec. 3.6). There are several reasons. As we shall soon see, the concept of the logarithm arises in the course of generalizing the theory of exponents (Sec. 2.13). Also, logarithms are extremely useful in performing such various computations as are often required in the solution of algebraic problems. Furthermore, in the next chapter, we consider several specific applications of logarithms.

16.2. THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS

In our previous work we dealt with algebraic expressions involving terms of the type x^n, where x is a variable called the *base* and n is a constant called the *exponent*. Now if we interchange the roles of the base and exponent, we obtain an expression which we may write in the form b^x, where b, the base, is a constant and x, the exponent, is a variable. Such an expression is called an *exponential function*.

In Sec. 2.13 we gave a meaning to b^x for all rational values of x. Thus, by the laws of exponents, $2^3 = 2 \cdot 2 \cdot 2$, $2^{-3} = 1/2^3$, and $2^{3/2} = \sqrt{2^3}$. But if x is irrational, no meaning has been assigned to b^x. For example, $2^{\sqrt{2}}$ has not been previously defined. We now proceed to generalize the laws of exponents by giving a meaning to b^x when x is irrational, and hence, for all *real* values of x.

To fix our ideas, let the exponent x be $\sqrt{2}$, an irrational number approximately equal to $1.41421 \cdots$. Then (Sec. 10.5) we *define* $\sqrt{2}$ as the limit of the sequence of rational numbers $1, 1.4, 1.41, 1.414, \cdots$. For each of these values, b^x takes on a corresponding value. It is shown in advanced treatises that if $b > 0$, this sequence of values of b^x approaches a limit, and this limit is *defined* as the value of $b^{\sqrt{2}}$. In general, if a is any real number,

$$\lim_{x \to a} b^x = b^a, \qquad b > 0. \tag{1}$$

The relation (1) means that a small change in x causes only a small change in the value of b^x; such a function is said to be *continuous*. Hence the graph of the exponential function

$$y = b^x, \qquad b > 0, \tag{2}$$

is a smooth continuous curve, as shown in Fig. 43. In this graph, $b > 1$. We shall see later that there are two particular values of the constant b which are of great importance, and both are greater than unity. The graph exhibits the following characteristics of the exponential function b^x when $b > 1$:

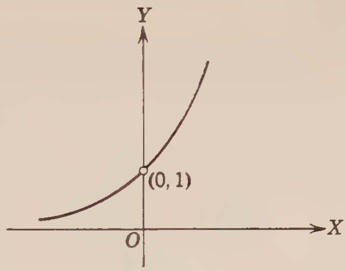


Figure 43

(a) Since the graph lies entirely above the X-axis, b^x is positive for every real value of x.

(b) b^x increases as x increases. As x increases without limit, b^x does likewise, and we write

$$\lim_{x \to \infty} b^x = \infty.$$

(c) For $x < 0$, $b^x < 1$; for $x = 0$, $b^x = 1$; for $x > 0$, $b^x > 1$.

(d) As x increases numerically without limit in the negative direction, b^x approaches zero, and we write

$$\lim_{x \to -\infty} b^x = 0.$$

In addition, we note the following two facts, which are established by advanced methods:

(1) If x is any real number, rational or irrational, and $b > 0$, the exponential function b^x obeys all the laws of exponents (Sec. 2.13).

(2) If $b > 0$, one and only one value of $y > 0$ corresponds to each real value of x in the relation $y = b^x$. We then say that b^x *is a single-valued function of* x. This fact is also illustrated by the graph in Fig. 43.

In relation (2), where y is expressed directly as a function of x, y may be obtained for particular rational values of b and x by algebraic operations. Thus, for $b = 2$ and $x = \frac{3}{2}$, $y = b^x = 2^{3/2} = \sqrt{2^3} = 2\sqrt{2}$. If x is irrational, y may be approximated, as we have seen, by using algebraic operations with rational values approximating x. Now consider the converse problem of finding x when b and y are given. For example, let us study the problem of finding x for the relation

$$5 = 2^x.$$

We can readily see in this case that x must lie between 2 and 3, for $2^2 = 4$ and $2^3 = 8$. It is evident that the value of x must be obtained by an approximating process. To meet this situation, we consider the inverse of the exponential function (2) and write it in the form

(3) $$x = \log_b y, \qquad b > 0,$$

which is read "x equals the logarithm of y to the base b." Since the two equations (2) and (3) represent exactly the same relation, we see that a logarithm is an exponent ; in fact, we have the following

Definition. The logarithm of a number to a given base is the exponent of the power to which the base must be raised in order to equal the number.

In view of the equivalence of equations (2) and (3), the graph in Fig. 43 also represents the logarithmic function defined by equation (3) when $b > 1$. Hence, for each point on the graph, the value of y represents a positive number and the corresponding value of x represents the logarithm of that number to the base b. Hence the characteristics of the exponential function may now be translated into the following corresponding properties of the logarithmic function:

(a) Only positive numbers have real logarithms. The logarithms of negative numbers do not exist in the real number system; it is shown in advanced work that such logarithms are complex numbers. The logarithm of zero is undefined.

(b) As a number y increases, its logarithm x also increases. As y increases without limit, so does x and we write

$$\lim_{y \to \infty} \log_b y = \infty.$$

(c) For $y < 1$, $\log_b y < 0$; for $y = 1$, $\log_b y = 0$; for $y > 1$, $\log_b y > 0$.

(d) As a number y approaches zero, its logarithm increases without limit in the negative direction, and we write

$$\lim_{y \to 0} \log_b y = -\infty.$$

Also, it is shown by advanced methods that if $b > 0$, the logarithmic function $\log_b y$ is single valued and continuous for all positive values of y. This is also confirmed by the graph in Fig. 43.

Since it is customary to have x represent the independent variable and y the dependent variable in a functional relation, it is usual to interchange the roles of x and y as given in relation (3) and to write the logarithmic relation in the form

(4) $$y = \log_b x, \qquad b > 0,$$

where x now represents the numbers and y the corresponding logarithms. The graph of equation (4) is shown in Fig. 44 and is the usual representation of the logarithmic function. Note that the graphs in Figs. 43 and 44 are identical in form; they differ only in their positions relative to the coordinate axes.

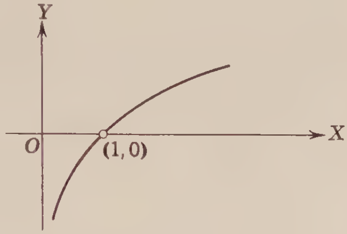


Figure 44

NOTE. Theoretically any real number except 0 or 1 may be used as the base b of a system of logarithms. Consider the relation $y = b^x$ and its equivalent form $x = \log_b y$.

If $b = 0$, $y = b^x = 0$ for all values of x except 0, in which case it is undefined. Also, if $b = 1$, $y = b^x = 1$ for all values of x. Hence neither 0 nor 1 can serve as the base of a system of logarithms.

If b is negative, $y = b^x$ may be negative or complex for certain values of x. The discussion of this case is beyond the scope of this book.

If b lies between 0 and 1, $y = b^x$ decreases as x increases. But in the systems of logarithms in actual use, we have chosen $y = b^x$ as a function which increases as x increases.

Hence, for simplicity and for practical purposes, we will take the base of a system of logarithms as a positive number greater than unity.

Example. In each of the following relations, find the value of the specified letter:

(a) If $x = \log_2 8$, find x.

(b) If $\log_b \frac{1}{16} = 4$, find b.

(c) If $\log_3 y = -2$, find y.

SOLUTION. In each case, we transform the given relation into its equivalent exponential form.

(a) From $x = \log_2 8$, we have the exponential relation $2^x = 8$, whence $x = 3$.

(b) From $\log_b \frac{1}{16} = 4$, we have the exponential relation $b^4 = \frac{1}{16}$, whence $b = \frac{1}{2}$.

(c) From $\log_3 y = -2$, we have the exponential relation $3^{-2} = y$, whence $y = \frac{1}{9}$.

EXERCISES. GROUP 58

In each of Exs. 1–6, translate the given relation into logarithmic form.

1. $2^4 = 16$. 2. $3^{-1} = \frac{1}{3}$. 3. $\left(\frac{1}{8}\right)^{2/3} = \frac{1}{4}$.

4. $N = b^x$. 5. $x^y = z$. 6. $u = v^w$.

In each of Exs. 7–12, translate the given relation into exponential form.

7. $\log_{10} 100 = 2$. 8. $\log_3 81 = 4$. 9. $\log_{10} 0.1 = -1$.

10. $\log_b a = c$. 11. $\log_8 4 = \frac{2}{3}$. 12. $\log_{\sqrt{2}} 1 = 0$.

In each of Exs. 13–16, evaluate the given logarithm.

13. $\log_{10} 1000$. 14. $\log_{10} 0.001$. 15. $\log_5 625$. 16. $\log_{0.2} 0.008$.

17. If $\log_b 0.01 = -2$, find b. 18. If $\log_7 N = 0$, find N.

19. If $\log_4 8 = x$, find x. 20. If $\log_b 9 = -2$, find b.

21. If $\log_4 N = 3$, find N.

22. Show that $\log_b 1 = 0$ and that $\log_b b = 1$.

23. Show that $\log_b b^x = x$ and that $b^{\log_b x} = x$.

In each of Exs. 24–26, find the inverse of the given function.

24. $y = 3^{-x}$. 25. $y = 10^{x-1}$. 26. $y = \log_{10} \frac{1}{x}$.

27. Show that the exponential function $y = b^x$ has the property that if x is given a sequence of values in arithmetic progression, the corresponding values of y are in geometric progression.

28. Plot the graph of the exponential curve $y = 2^x$.

29. Plot the graph of the exponential curve $y = (\frac{1}{2})^x$. Compare the result with the graph obtained in Ex. 28.

30. Plot the graph of the exponential curve $y = 3^{-x}$. Contrast this graph with the graph in Fig. 43.

31. List the characteristics of the graph obtained in Ex. 29 and contrast them with those obtained for the graph in Fig. 43.

32. Plot the graph of the logarithmic curve $y = \log_2 x$ by using the equivalent exponential relation.

33. Plot the graph of the logarithmic curve $y = \log_{1/2} x$ by using the equivalent exponential relation, and compare the result with that obtained in Ex. 32.

34. List the characteristics of the logarithmic function whose graph is given in Fig. 44.

35. List the characteristics of the graph obtained in Ex. 33 and contrast them with those obtained in Ex. 34.

16.3. FUNDAMENTAL PROPERTIES OF LOGARITHMS

We have seen that a logarithm is an exponent. Hence, by translating the laws of exponents into logarithmic form, we obtain the laws of logarithms.

We will now establish four fundamental theorems on logarithms which are the result, respectively, of the following four laws of exponents (Sec. 2.13):

(1) $$b^x \cdot b^y = b^{x+y}.$$

(2) $$b^x \div b^y = b^{x-y}.$$

(3) $$(b^x)^n = b^{nx}.$$

(4) $$\sqrt[n]{b^x} = b^{x/n}.$$

In the theorems which follow we are concerned with three positive numbers M, N, and b. We may therefore write

(5) $$M = b^x \text{ and } N = b^y,$$

whence

(6) $$x = \log_b M \text{ and } y = \log_b N.$$

Theorem 1. *The logarithm of the product of two positive numbers is equal to the sum of the logarithms of those numbers, that is,*

$$\log_b MN = \log_b M + \log_b N.$$

PROOF. From (5) and (1) we have

$$MN = b^x \cdot b^y = b^{x+y}$$

whence, from the definition of logarithm and (6),

$$\log_b MN = x + y = \log_b M + \log_b N.$$

This theorem may be readily extended to the product of three or more positive numbers.

Theorem 2. *The logarithm of the quotient of two positive numbers is equal to the logarithm of the dividend minus the logarithm of the divisor, that is,*

$$\log_b \frac{M}{N} = \log_b M - \log_b N.$$

PROOF. From (5) and (2) we have

$$\frac{M}{N} = \frac{b^x}{b^y} = b^{x-y}$$

whence, from the definition of logarithm and (6),

$$\log_b \frac{M}{N} = x - y = \log_b M - \log_b N.$$

Theorem 3. *The logarithm of the nth power of a positive number is equal to n times the logarithm of the number, that is,*

$$\log_b M^n = n \log_b M.$$

PROOF. From (5) and (3) we have

$$M^n = (b^x)^n = b^{nx}$$

whence, from the definition of logarithm and (6),

$$\log_b M^n = nx = n \log_b M.$$

Theorem 4. *The logarithm of the real positive nth root of a positive number is equal to the logarithm of the number divided by n, that is,*

$$\log_b M^{1/n} = \frac{1}{n} \log_b M.$$

PROOF. From (5) and (4) we have

$$M^{1/n} = \sqrt[n]{b^x} = b^{x/n}$$

whence, from the definition of logarithm and (6),

$$\log_b M^{1/n} = \frac{x}{n} = \frac{\log_b M}{n}.$$

We list here, also, the following important properties of a logarithm, which are a direct consequence of the definition of a logarithm.

(7) $$\log_b b = 1.$$

(8) $$\log_b b^n = n.$$

(9) $$b^{\log_b N} = N.$$

The logarithm of a number depends upon the base. The logarithm of a positive number to any base $a > 0$ may be expressed in terms of logarithms to another base $b > 0$ by means of the following theorem.

Theorem 5. *The logarithm of a positive number N to the base a is equal to the logarithm of N to another base b divided by the logarithm of a to the base b, that is,*

$$\log_a N = \frac{\log_b N}{\log_b a}.$$

PROOF. Let $\log_a N = x$
so that

$$N = a^x.$$

Taking the logarithms of both sides to the base b, we have, by Theorem 3,

$$\log_b N = x \log_b a$$

whence

$$x = \frac{\log_b N}{\log_b a}, \text{ or}$$

(10)
$$\log_a N = \frac{\log_b N}{\log_b a},$$

the required relation.

If we set $N = b$ in (10) we have, by (7), the following relation:

(11)
$$\log_a b = \frac{1}{\log_b a}.$$

NOTES. 1. The relation (10) of Theorem 5 for a *change of base* is useful when we wish to obtain the logarithm of any number to some base a and the available table of logarithms is to another base b.

2. In relation (11), the number $\log_a b$ is called the *modulus* of the system of logarithms to the base a with respect to the system of logarithms to the base b.

We shall see later that the results of Theorems 1 through 4 are extremely useful in performing arithmetical computations involving the operations of multiplication, division, involution, and evolution. But, for the present, we shall restrict their use to exponential and logarithmic expressions, as shown in the following examples.

Example 1. Find the inverse of the function $y = \dfrac{b^x - b^{-x}}{2}$, $b > 0$.

SOLUTION. We are required to solve the equation

$$y = \frac{b^x - b^{-x}}{2}$$

for x in terms of y.

Multiplying by $2b^x$, we have

$$2yb^x = b^{2x} - 1.$$

Rearranging terms,
$$b^{2x} - 2yb^x - 1 = 0.$$

This last equation is quadratic in form (Sec. 5.6), for, if we let $z = b^x$, the equation may be written as

$$z^2 - 2yz - 1 = 0.$$

Hence, solving for z or b^x by the quadratic formula (Sec. 5.4), we have

(12)
$$b^x = \frac{2y \pm \sqrt{4y^2 + 4}}{2} = y \pm \sqrt{y^2 + 1}.$$

Now $\sqrt{y^2 + 1} > y$, and since the exponential function b^x is always positive (Sec. 16.2), we discard the minus sign in (12) and write

$$b^x = y + \sqrt{y^2 + 1}$$

whence we have our required inverse,

$$x = \log_b (y + \sqrt{y^2 + 1}).$$

Example 2. Find the inverse of the function $y = \log_b x - \log_b (1 + x)$.

SOLUTION. By Theorem 2, the given function may be written in the form

$$y = \log_b \frac{x}{1 + x}$$

whence
$$b^y = \frac{x}{1 + x}$$

and
$$b^y + b^y x = x.$$

Then
$$b^y = x(1 - b^y)$$

and
$$x = \frac{b^y}{1 - b^y}.$$

Example 3. Show that

$$\log_b (\sqrt{x + 2} - \sqrt{x + 1}) = -\log_b (\sqrt{x + 2} + \sqrt{x + 1}).$$

SOLUTION. Since we are to obtain a result involving $\sqrt{x + 2} + \sqrt{x + 1}$, we introduce this expression as follows:

$$\sqrt{x + 2} - \sqrt{x + 1} = (\sqrt{x + 2} - \sqrt{x + 1}) \cdot \frac{\sqrt{x + 2} + \sqrt{x + 1}}{\sqrt{x + 2} + \sqrt{x + 1}}$$

$$= \frac{x + 2 - (x + 1)}{\sqrt{x + 2} + \sqrt{x + 1}} = \frac{1}{\sqrt{x + 2} + \sqrt{x + 1}}.$$

Hence, $\log_b (\sqrt{x + 2} - \sqrt{x + 1}) = \log_b \dfrac{1}{\sqrt{x + 2} + \sqrt{x + 1}}$

By Theorem 2, $\qquad = \log_b 1 - \log_b (\sqrt{x + 2} + \sqrt{x + 1})$

By Property (8) for $n = 0$, $\qquad = -\log_b (\sqrt{x + 2} + \sqrt{x + 1}).$

EXERCISES. GROUP 59

1. Extend Theorem 1 (Sec. 16.3) to the product of three or more positive numbers.

2. Show that the logarithm of the geometric mean of two positive numbers is equal to the arithmetic mean of their logarithms.

3. Obtain the result of Theorem 4 directly from Theorem 3 (Sec. 16.3).

4. Obtain property (8) of Sec. 16.3 from Theorem 3 and property (7).

5. Obtain property (7) of Sec. 16.3 from property (8).

6. Obtain property (9) of Sec. 16.3 by the following method: Set $b^{\log_b N} = y$ and take the logarithms of both sides to the base b.

7. If N, a, and b are positive numbers, show that $\log_b N = \log_a N \cdot \log_b a$.

8. Show that $\log_b N^{-n} = -n \log_b N$.

In each of Exs. 9–14, express the given logarithm in terms of simpler logarithms.

9. $\log_b \dfrac{x^2 - 1}{x^2 - 4}$.

10. $\log_b \dfrac{x^2}{x^3 + 1}$.

11. $\log_b \dfrac{x(x + 2)^2}{(x - 2)^4}$.

12. $\log_b \dfrac{\sqrt{x^2 + 1}}{3x^2}$.

13. $\log_b \sqrt{\dfrac{x^2 + 1}{x^2 + 2}}$.

14. $\log_b \sqrt{\dfrac{x(x^2 - 5)}{(x^2 + 3)(x^2 - 3)}}$.

In each of Exs. 15–18, find the value of x.

15. $\log_b x = \log_b 2 + 3 \log_b 2 - \log_b 4$.

16. $\log_b x = \frac{1}{2} \log_b 3 + \log_b 4 - \frac{1}{2} \log_b 2$.

17. $\log_{10} x = 2 \log_{10} 3 + 3 \log_{10} 2 - 2$.

18. $\log_{10} x = \frac{1}{2} \log_{10} 16 - \frac{1}{3} \log_{10} 8 + 1$.

19. Simplify: (a) $b^{\log_b 3}$; (b) $b^{2 \log_b 2}$.

20. Simplify: (a) $10^{\frac{1}{2} \log_{10} 8}$; (b) $10^{3 \log_{10} 2}$.

In each of Exs. 21–30, find the inverse of the given function.

21. $y = b^{x+2}$.

22. $y = b^{\frac{x-1}{x}}$.

23. $y = \dfrac{1}{1 - b^x}$.

24. $y = \dfrac{b^x - 1}{b^x + 1}$.

25. $y = \dfrac{1 + b^x}{1 - b^x}$.

26. $y = \dfrac{b^x + b^{-x}}{2}$.

27. $y = \log_b \dfrac{x}{x - 1}$.

28. $y = \log_b \dfrac{x}{x^2 + 1}$.

29. $y = \log_b \dfrac{1 \pm \sqrt{1 - x^2}}{x}$.

30. $y = \log_b \dfrac{1 + \sqrt{1 + x^2}}{x}$.

31. Show that $\dfrac{1}{2} \log_b \dfrac{3 + 2\sqrt{2}}{3 - 2\sqrt{2}} = \log_b (3 + 2\sqrt{2})$.

32. Show that $\log_b (\sqrt{x + 3} + \sqrt{x + 2}) = -\log_b (\sqrt{x + 3} - \sqrt{x + 2})$.

33. Show that $\log_b \dfrac{\sqrt{a^2 + x^2} + a}{x} = -\log_b \dfrac{\sqrt{a^2 + x^2} - a}{x}$.

34. Show that $\log_b (x \pm \sqrt{x^2 - 1}) = \pm\log_b (x + \sqrt{x^2 - 1})$.

35. Show that $\log_b (1 - \sqrt{1 - x^2}) = 2 \log_b x - \log_b (1 + \sqrt{1 - x^2})$.

16.4. SYSTEMS OF LOGARITHMS

We have previously seen that it is both theoretically and practically desirable that the base of a system of logarithms be positive and greater than unity. Two such bases are in common use, the number 10 and an irrational number usually denoted by e and approximately equal to $2.71828 \cdots$.

The system of logarithms with the base 10 is called the *common* or *Briggs' system*; it is generally used for ordinary arithmetical computations. The system of logarithms with the base e is called the *natural* or *Napierian system*; it is used almost exclusively in the calculus and advanced mathematics.

Later we shall see that the common system of logarithms with the base 10 has definite advantages in computations involving the numbers of our decimal system. But we are not now in a position to show the advantages of the base e; later, in the calculus, the student will readily see the convenience of natural logarithms whose base e is *defined* as the following limit:

$$e = \lim_{z \to \infty} \left(1 + \frac{1}{z}\right)^z = 2.71828 \cdots.$$

The relation between common and natural logarithms may be obtained by means of Theorem 5 (Sec. 16.3), where it is shown that for any positive number N and two different bases a and b,

$$\log_a N = \frac{\log_b N}{\log_b a}.$$

In this relation, let $a = e$ and $b = 10$, so that

(1) $$\log_e N = \frac{\log_{10} N}{\log_{10} e}$$

from which

(2) $$\log_{10} N = \log_{10} e \cdot \log_e N.$$

From a table of logarithms, we may find that

$$\log_{10} e = 0.4343$$

whence its reciprocal

$$\frac{1}{\log_{10} e} = \frac{1}{0.4343} = 2.3026.$$

Hence, relations (1) and (2) may be written in the respective forms

$$\log_e N = 2.3026 \log_{10} N,$$
$$\log_{10} N = 0.4343 \log_e N.$$

The number $\log_{10} e = 0.4343$ is called the *modulus* of common logarithms with respect to natural logarithms. Then by relation (11) and Note 2 of Sec. 16.3, the reciprocal of $\log_{10} e$ or $\log_e 10 = 2.3026$ is called the *modulus* of natural logarithms with respect to common logarithms.

Since we shall, in general, use only the bases 10 and e, we may for convenience omit these bases by adopting a simple convention. Thus, for the logarithm of a number N to the base 10, we shall write $\log N$ instead of $\log_{10} N$. Also, for the logarithm of N to the base e, we shall write $\ln N$ instead of $\log_e N$. The term $\ln N$ is also read "natural log of N." For example, relation (2) above may be written

$$\log N = \log e \cdot \ln N.$$

16.5. EXPONENTIAL EQUATIONS

An equation in which a variable appears as an exponent is called an *exponential equation*. Thus, $2^{x+1} = 8$ and $e^x - e^{-x} = 1$ are examples of exponential equations.

To effect the solution of an exponential equation, the equation should first be solved, if necessary, for the exponential function. The next step is to take logarithms of both members to an appropriate base. In this step we use the fact that if two expressions are equal, their logarithms are equal since, as we have seen (Sec. 16.2), the exponential function and its inverse, the logarithmic function, are single valued. The process is best explained by means of examples where it is important to remember that the exponential function is always positive and that we are considering only real values.

Example 1. Solve the equation

$$e^x - e^{-x} = 1.$$

SOLUTION. Multiplying through by e^x, we obtain

$$e^{2x} - 1 = e^x$$

or

$$e^{2x} - e^x - 1 = 0.$$

This equation is quadratic in form (Sec. 5.6) when we consider e^x as the variable. Hence, solving for e^x by the quadratic formula, we obtain

$$e^x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

Since e^x is always positive, we discard the minus sign and write

$$e^x = \frac{1 + \sqrt{5}}{2}.$$

Taking logarithms to the base e, we obtain

$$x = \ln \frac{1 + \sqrt{5}}{2}$$

as the required solution.

Example 2. Solve the equation

$$e^{3x} - 2e^{2x} - 2e^{x} - 3 = 0.$$

SOLUTION. If we let $y = e^x$ in this equation, it takes the form

$$(1) \qquad y^3 - 2y^2 - 2y - 3 = 0,$$

a rational integral equation which may be solved by the methods of Chapter 11. Thus we find that $y = 3$ is a root of equation (1). Removing this root by synthetic division, we obtain the depressed equation

$$y^2 + y + 1 = 0,$$

which has no real roots.

Since e^x must be positive, the only value for y is 3. Hence $e^x = 3$ whence $x = \ln 3$ is the required solution.

Example 3. Solve the following equation for t:

$$I = \frac{E}{R}(1 - e^{-\frac{Rt}{L}}).$$

SOLUTION. We first isolate the exponential function $e^{-\frac{Rt}{L}}$. Thus, multiplying by R, we have

$$IR = E - Ee^{-\frac{Rt}{L}}$$

whence

$$IR - E = -Ee^{-\frac{Rt}{L}}$$

and

$$e^{-\frac{Rt}{L}} = \frac{E - IR}{E}.$$

Taking logarithms to the base e, we obtain

$$-\frac{Rt}{L} = \ln\left(\frac{E - IR}{E}\right)$$

whence

$$t = -\frac{L}{R}\ln\left(\frac{E - IR}{E}\right).$$

16.6. LOGARITHMIC EQUATIONS

An equation containing one or more logarithmic functions of one or more variables is called a *logarithmic equation.* Thus,

$$\log(x - 2) + \log(x + 1) + 1 = \log 40$$

and

$$2 \ln y = 3 \ln(x - 1) + x$$

are examples of logarithmic equations.

The solution of a logarithmic equation involving a single variable may be effected by first transforming it into a relation free of logarithms. In this process we use the fact that if the logarithms of two expressions are equal, the expressions themselves are equal. Also, it is very important to check all solutions since we do not consider those values of the variable which lead to the logarithms of negative quantities. We illustrate the procedure in the following example.

Example 1. Solve the equation

$$\log(x - 2) + \log(x + 1) + 1 = \log 40. \tag{1}$$

SOLUTION. Since the logarithms here are to the base 10, we replace 1 by log 10 so that we may write

$$\log(x - 2) + \log(x + 1) + \log 10 = \log 40$$

whence, by Theorem 1 (Sec. 16.3), we have

$$\log 10\,(x - 2)(x + 1) = \log 40.$$

Hence,

$$10\,(x - 2)(x + 1) = 40,$$

$$x^2 - x - 2 = 4,$$

and

$$x^2 - x - 6 = 0.$$

The solution of this last equation is readily found to give two values of x, namely, -2 and 3. But we must reject the solution -2 since substitution of this value in (1) results in the logarithms of negative numbers. But 3 is a valid solution since substitution in (1) gives us

$$\log 1 + \log 4 + 1 = \log 40$$

whence

$$0 + \log 4 + 1 = \log 4 + \log 10$$

or

$$\log 4 + 1 = \log 4 + 1.$$

We next consider a logarithmic equation involving more than one variable.

Example 2. Transform the following equation into one which is free of logarithms:

$$2 \ln y = 3 \ln (x - 1) + x.$$

SOLUTION. Since the given equation involves natural logarithms, we replace x by $\ln e^x$ and obtain

$$2 \ln y = 3 \ln (x - 1) + \ln e^x.$$

Then by the properties of logarithms (Sec. 16.3), we have

$$\ln y^2 = \ln (x - 1)^3 e^x$$

whence

$$y^2 = (x - 1)^3 e^x,$$

the required equation.

EXERCISES. GROUP 60

1. If N is any positive number, show that $\ln N = \ln 10 \cdot \log N$.

2. Show that $\log e = \dfrac{1}{\ln 10}$ and that $\ln 10 = 2.3026$.

3. Plot the curve $y = e^{-x^2}$. A fair approximation of the shape of this curve may be obtained by taking $e = 3$. It is known as a *probability curve* (Sec. 14.6).

4. Work out all the details of Example 2 of Sec. 16.5.

In each of Exs. 5–20, solve the given equation.

5. $3^{x+1} = 81$. 6. $2^{x-1} = 16$. 7. $5^{x^2+x} = 25$.

8. $2^{x+2} = 4^{x-1}$. 9. $5^{x+1} = 3^{2x}$. 10. $7^x = 2^{2x+1}$.

11. $e^x - e^{-x} = 2$. 12. $e^x + e^{-x} = 1$.

13. $e^{2x} - 2e^x - 3 = 0$. 14. $e^{2x} + 5e^x + 6 = 0$.

15. $e^{2x} - 2e^{-2x} - 1 = 0$. 16. $2e^{3x} - 4e^{-3x} - 7 = 0$.

17. $e^{3x} - 3e^{2x} + 4e^x - 4 = 0$. 18. $e^{2x} - 2e^x - 5 + 6e^{-x} = 0$.

19. $2e^{4x} + e^{3x} + e^{2x} + 11e^x - 6 = 0$.

20. $3e^{3x} - 7e^{2x} - 19e^x - 5 + 4e^{-x} = 0$.

21. In geometric progression (Sec. 10.3), we have the relation $a_n = a_1 r^{n-1}$. Solve this relation for n in terms of a_1, a_n, and r.

22. In geometric progression (Sec. 10.3), we have the relation $s_n = \dfrac{a_1(1 - r^n)}{1 - r}$. Solve this relation for n in terms of a_1, s_n, and r.

23. An electric circuit containing resistance and capacitance in series has an equation of the form $Q = CE(1 - e^{-t/CR})$. Solve this equation for t.

24. In compound interest, the amount A and the principal P are connected by the relation $A = P(1 + r)^n$. Solve this relation for n in terms of A, P, and r.

In each of Exs. 25–33, solve the given equation.

25. $\log x - \log (x - 2) = \log 2$. 26. $\log x + \log (x - 1) = \log 6$.

27. $\ln 12 - \ln (x - 1) = \ln (x - 2)$.

28. $\log(x - 2) + \log(x - 3) = \log 2$.

29. $\log(x + 2) + \log(x - 1) = 1$. 30. $\log(2x - 3) = 1 - \log(x - 2)$.

31. $\log(3x + 1) = 2 - \log(x + 7)$.

32. $\log(x + 1) + \log(x - 2) = 1 - \log(x - 3)$.

33. $2\log(x + 3) + \log(x + 2) = 2$.

In each of Exs. 34–40, transform the given equation into one which is free of logarithms.

34. $\log x + \log y = \log 4$. 35. $\ln(x + y) + \ln(x - y) = 0$.

36. $2\log y - x = \log x$. 37. $3\ln x - 2\ln y = 1$.

38. $\log(x + y) - \log x - \log y = \log 3 - \log(x^2 - xy + y^2)$.

39. $2\ln 2x - \ln(z + 2y) = \ln(z - 2y)$.

40. $\ln x + 2\ln y - x - y = z - 3\ln z$.

16.7. TABLES OF LOGARITHMS

Extensive tables of logarithms to both the base 10 and the base *e* have been constructed. These logarithms are computed by means of certain infinite series, and the determination is explained in the calculus. For our purposes here, we consider the manner of using a logarithmic table.

In tables of natural logarithms, the actual logarithm is given for each number listed. But this is not true for tables of common logarithms where only part of the logarithm is given for each number listed. It becomes necessary, therefore, to explain the manner of using a table of common logarithms. A short table of common logarithms is given in Appendix II and is the table of reference in this and the following section.

To fix our ideas, we first set up a skeleton table of common logarithms as shown. This table exhibits the properties of logarithms previously discussed in Sec. 16.2. We note, for example, that the logarithms of all the positive numbers exhaust the entire system of real numbers, thus excluding the logarithms of negative numbers from the real number system.

x	$\log x$
∞	∞
$\uparrow$	$\uparrow$
1000	3
100	2
10	1
1	0
0.1	-1
0.01	-2
0.001	-3
$\downarrow$	$\downarrow$
0	$-\infty$

It is evident that integral powers of 10 are the only numbers whose common logarithms are integers. Every other number, therefore, has as its logarithm an integer plus or minus a fraction which is expressed as a decimal correct to a certain number of places. For example, the logarithm of 225 is 2.3522, correct to 4 decimal places.

Since the logarithm of a number increases as the number increases, we can readily determine the successive integers between which the common logarithm of a number lies. Thus, for a number between 1 and 10, the logarithm lies between 0 and 1; for a number between 10 and 100, it lies between 1 and 2, and so forth. Also, for a number between 0.1 and 1, the logarithm lies between 0 and -1; for a number between 0.1 and 0.01, it lies between -1 and -2. But the decimal portion of a logarithm cannot be determined by inspection, and it is precisely this information which a table of common logarithms provides.

The logarithm of a number between 100 and 1000 lies between 2 and 3 and is therefore equal to 2 plus a decimal. The logarithm of a number between 0.01 and 0.001 lies between -2 and -3; it is therefore either equal to -2 minus a decimal or else -3 plus a decimal. In this latter case we elect to take the logarithm as -3 plus a decimal. In general, for any number, the decimal portion of its logarithm is always taken as positive (or zero); this convention, as we shall see, has a definite advantage in extending the range of a table of common logarithms.

Summarizing, a common logarithm consists of the sum of two parts, an integer and a positive (or zero) decimal fraction. The integer, which can be positive, negative, or zero, is called the *characteristic* and may be obtained by inspection as described in the rule below. The decimal fraction is called the *mantissa* and is obtained from a table of common logarithms.

A *rule for obtaining the characteristic* of the logarithm of a number N is as follows:

(1) If $N \geq 1$, the characteristic of $\log N$ is 1 less than the number of digits in N to the left of the decimal point.

(2) If $N < 1$ and is written in decimal form, the characteristic of $\log N$ is negative and is numerically 1 more than the number of zeros immediately to the right of the decimal point.

As an illustration of this rule, we may note that the logarithms of the numbers 4232, 321.3, 85.72, 1.26, 0.843, 0.0436, and 0.002917 have the respective characteristics 3, 2, 1, 0, -1, -2, and -3.

We shall now show the advantage of having a non-negative mantissa. Let b be a positive number lying in the range $1 \leq b < 10$. Then any positive number N may be written in the form

$$N = b \cdot 10^n, \tag{1}$$

where n is a positive or negative integer or zero. Thus,

$$4232 = 4.232 \times 10^3,$$

$$1.26 = 1.26 \times 10^0,$$

$$0.0436 = 4.36 \times 10^{-2}, \text{ and so forth.}$$

Note that the significant digits in b are precisely the same sequence of significant digits in N.

From relation (1) we have

(2) $$\log N = \log b + n.$$

The characteristic of $\log b$ is zero; let its mantissa be represented by m so that $\log b = m$. Then relation (2) may be written in the form

(3) $$\log N = n + m,$$

where n is the characteristic and m is the mantissa. Note that while n may vary in accordance with the magnitude of N, the mantissa m remains the same as that in the logarithm of b. Because of the importance of this result we record it as

Theorem 6. *If two positive numbers have the same sequence of significant digits but differ in magnitude, their respective logarithms have different characteristics but precisely the same mantissa.*

As an illustration of Theorem 6, we have

$$\log 1.42 = 0.1523,$$
$$\log 1420 = \log (1.42) \cdot 10^3 = (\log 1.42) + 3 = 3.1523,$$
$$\log 0.142 = \log (1.42) \cdot 10^{-1} = (\log 1.42) - 1 = \bar{1}.1523,$$
$$\log 0.00142 = \log (1.42) \cdot 10^{-3} = (\log 1.42) - 3 = \bar{3}.1523.$$

In the case of a logarithm whose characteristic is negative, we write the minus sign above the characteristic to show that it alone is negative, the mantissa as usual being positive. Thus, since 0·142 is less than unity, its logarithm is negative as we can see by writing

$$\bar{1}.1523 = -1 + 0.1523 = -0.8477.$$

In order to avoid negative characteristics, it is common practice to add 10 to the characteristic and then subtract 10 at the right of the mantissa. Thus, the logarithm $\bar{1}.1523$ is then written as $9.1523 - 10$. We will, however, indicate a negative characteristic by writing the minus sign above it.

Having discussed the determination of the characteristic, it remains to show how the mantissa is obtained from a table of logarithms, such as appears in Appendix II. If the given number has three or less significant digits, we locate the first two digits in the left column and the third digit at the top of the table. Our required mantissa is then the four digit entry located in the same row as the first two digits and in the same column as the third digit. Thus, for the number 142, the first two digits appear in the left column in the fifth row, and the third digit 2 at the top in the third column. The corresponding entry is found to be 1523; hence the mantissa of log 142 is 0.1523. As practice in the use of the table, the student should verify the following logarithms: $\log 34.5 = 1.5378$, $\log 456 = 2.6590$, $\log 2.03 = 0.3075$, $\log 0.075 = \bar{2}.8751$.

If the given number has four or more significant digits, the mantissa of its logarithm does not appear in the table but may be obtained approximately by the method of *linear interpolation* discussed in Sec. 11.10. The method here is based on the assumption that for a small change in a number, the change in its logarithm is proportional to the change in the number. The procedure is best explained by means of an example.

Example 1. Find the logarithm of 1424.

SOLUTION. The characteristic is, of course, 3. The mantissa lies between the mantissa of 1420 and the mantissa of 1430. From the table we have

$$\text{mantissa of } 1430 = 0.1553,$$
$$\text{mantissa of } 1420 = 0.1523.$$

The difference between these two mantissas is 0.0030 and is called the *tabular difference*. The increase in the number from 1420 to 1430 is 10 and causes an increase in the mantissa of 0.0030. Hence, by proportion, an increase in the number from 1420 to 1424 or 4 causes an increase in the mantissa of $4/10 \times 0.0030$ or 0.0012. Hence the required mantissa is $0.1523 + 0.0012 = 0.1535$, and $\log 1424 = 3.1535$.

For further practice, the student should verify the following logarithms: $\log 5026 = 3.7012$, $\log 0.006241 = \bar{3}.7953$, $\log 8.325 = 0.9204$.

We next consider the inverse problem, namely, given the logarithm of a number, to find the number, appropriately called the *antilogarithm*. If the mantissa of the given logarithm appears exactly in the table, the significant figures of the antilogarithm may be obtained immediately; otherwise interpolation is necessary.

Example 2. Find the antilogarithm of (a) 1.9047; (b) $\bar{2}.6144$.

SOLUTION. (a) The mantissa 0.9047 appears exactly in the table in the row corresponding to 80 in the left column and in the column headed by the third digit 3. Hence the significant digits are 803 and the required antilogarithm is 80.3.

(b) The mantissa 0.6144 does not appear exactly in the table but lies between the consecutive mantissas 0.6138 and 0.6149 corresponding, respectively, to 4110 and 4120. Hence we have

Mantissa	*Number*
0.6149	4120
0.6138	4110

The tabular difference between the mantissas is 0.0011 due to a change of 10 in the number from 4110 to 4120. Hence, by proportion, an increase in the mantissa from 0.6138 to 0.6144 or 0.0006 causes an increase in the number of $6/11 \times 10 = 5$ very nearly. Therefore the required sequence of significant digits is $4110 + 5 = 4115$ and the required antilogarithm is 0.04115.

For further practice, the student should verify the following: antilog of $\bar{1}.6791 = 0.4777$; antilog of $2.8024 = 634.4$.

The use of a table of common logarithms enables us to obtain the logarithm of a number to any base by means of Theorem 5 (Sec. 16.3), where it is shown that

$$\log_a N = \frac{\log_b N}{\log_b a}. \tag{4}$$

The procedure is illustrated in

Example 3. Find $\log_6 0.86$.

SOLUTION. By relation (4) above,

$$\log_6 0.86 = \frac{\log 0.86}{\log 6} = \frac{\bar{1}.9345}{0.7782} = \frac{-1 + 0.9345}{0.7782}$$

$$= \frac{-0.0655}{0.7782} = -0.0842.$$

NOTE. More extensive tables of logarithms give the mantissas to five or more places and for a greater range of numbers than our own short table. The use of such tables results in greater accuracy and ease in computations. These tables usually give tabular differences and tables of proportional parts to facilitate interpolation.

16.8. LOGARITHMIC COMPUTATION

We now study the advantages of common logarithms in reducing the labor of arithmetical computations. In accordance with the properties of logarithms established in Sec. 16.3, it is possible to replace the operations of multiplication, division, involution, and evolution by the simpler operations of addition, subtraction, multiplication, and division, respectively. The procedure is best explained by means of examples.

Example 1. Evaluate $x = \dfrac{346 \times 0.0269}{45.21}$.

SOLUTION. By Theorems 1 and 2 (Sec. 16.3), we may write

$$\log x = \log 346 + \log 0.0269 - \log 45.21. \tag{1}$$

For actual computation we arrange the work as shown below, inserting the values of the logarithms obtained from the table of common logarithms in Appendix II.

$$\begin{aligned} \log 346 &= 2.5391 \\ +\log 0.0269 &= \bar{2}.4298 \\ \hline \log 346 + \log 0.0269 &= 0.9689 \\ -\log 45.21 &= -1.6552 \\ \hline \log x &= \bar{1}.3137 \end{aligned}$$

whence $x = 0.2059,$

where x is obtained as the antilogarithm of $\log x$.

In performing these operations the student must remember that the decimal portion or mantissa of a logarithm is always positive and that a negative characteristic has the minus sign written above it.

Example 2. Evaluate (a) $x = (0.162)^5$; (b) $\sqrt[3]{-0.085}$.

SOLUTION. (a) By Theorem 3 (Sec. 16.3),

$$\log x = 5 \log (0.162) = 5(\bar{1}.2095).$$

In performing this multiplication the student must be careful about the signs. The operation is actually as follows:

$$5(\bar{1}.2095) = 5(-1 + 0.2095) = -5 + 1.0475 = \bar{4}.0475.$$

That is, $\log x = \bar{4}.0475$

whence $x = 0.0001116.$

(b) Here the required cube root is a negative number. But we proceed with this operation as if all quantities are positive and then attach the proper (negative) sign to the result. Thus, let $y = \sqrt[3]{0.085}$. Then by Theorem 4 (Sec. 16.3),

$$\log y = \tfrac{1}{3} \log 0.085 = \tfrac{1}{3}(\bar{2}.9294).$$

The characteristic cannot be fractional and the mantissa must be positive. Hence, to perform this division by 3, we make the characteristic of log 0.085 a multiple of 3 by subtracting 1 and then add 1 to the mantissa. The actual operation is as follows:

$$\tfrac{1}{3}(\bar{2}.9294) = \tfrac{1}{3}(-3 + 1.9294) = -1 + 0.6431 = \bar{1}.6431.$$

That is, $\log y = \bar{1}.6431$

whence $y = 0.4396$

and $x = -0.4396.$

Some of the steps in the solutions of Examples 1 and 2 have been included for expository reasons but may be excluded in actual computations. Thus, relation (1) in the solution of Example 1 may be omitted; its significance is clearly indicated in the arrangement of the logarithmic work.

For speed and accuracy in computations it is desirable to make a tabular arrangement or scheme *before* looking up any logarithms. Then all logarithms may be entered at one time. The procedure is illustrated in the next example.

Example 3. Evaluate $x = \left(\dfrac{8264 \times 0.311}{2.351 \times 28.6}\right)^{1/2}$.

SOLUTION. If N represents the numerator and D the denominator of the fraction, the schematic arrangement of the logarithmic work appears as follows:

$$\begin{aligned} \log 8264 &= \\ +\log 0.311 &= \underline{\qquad\quad} \\ \log N &= \\ -\log D &= \underline{\qquad\quad} \\ \log N/D &= \\ \tfrac{1}{2}\log N/D &= \qquad\quad = \log x \\ x &= \end{aligned} \qquad \begin{aligned} \log 2.351 &= \\ +\log 28.6 &= \underline{\qquad\quad} \\ \log D &= \end{aligned}$$

The next step is to enter all the logarithms in the above arrangement and complete the operation, which then appears as follows:

$$\begin{aligned} \log 8264 &= 3.9172 \\ +\log 0.311 &= \underline{\bar{1}.4928} \\ \log N &= 3.4100 \\ -\log D &= \underline{1.8277} \\ \log N/D &= 1.5823 \\ \tfrac{1}{2}\log N/D &= 0.79115 = \log x \\ x &= 6.182. \end{aligned} \qquad \begin{aligned} \log 2.351 &= 0.3713 \\ +\log 28.6 &= \underline{1.4564} \\ \log D &= 1.8277 \end{aligned}$$

NOTE. For greater compactness in the arrangement of the logarithmic work involving the operation of division, some authors use cologarithms. The *cologarithm* of a number is the logarithm of its reciprocal. Its use permits addition in place of subtraction of logarithms. The abbreviation for cologarithm is colog. Thus, colog $N = \log 1/N$.

EXERCISES. GROUP 61

In each of Exs. 1–8, evaluate the given logarithm.

1. $\log_2 20$. 2. $\log_5 17$. 3. $\log_6 8.1$. 4. $\log_7 5$.
5. $\ln 3$. 6. $\ln 7$. 7. $\ln 10.3$. 8. $\log_4 2.31$.

9. Verify all the computations in Example 3 (Sec. 16.8).

10. Show that colog $N = -\log N$ and that $\log N/D = \log N +$ colog D.

In each of Exs. 11–26, evaluate the given expression by logarithms.

11. 431×0.4126.
12. $3.063 \div 28.41$.
13. $21.2 \times 13.11 \times 0.0061$.
14. $85.23 \times 61.4 \div 21.36$.
15. $\dfrac{7.203 \times 34.2}{85.11}$.
16. $\dfrac{3.87 \times 3.142}{2.718 \times 0.0116}$.
17. $(4.21)^{2/3}(0.7321)$.
18. $(21.39)^{1/4}(1.237)$.
19. $\dfrac{22.3 \times 0.041 \times 236.8}{521.3 \times 0.0026}$.
20. $\dfrac{181.2 \times 415.3 \times 62.91}{2013 \times 341.9 \times 85.86}$.
21. $\dfrac{(91.6)^2 \times \sqrt[3]{41.62}}{\sqrt[5]{724.1}}$.
22. $\dfrac{\sqrt{32.17} \times \sqrt[3]{55.6}}{\sqrt[4]{5.113} \times \sqrt{86.92}}$.
23. $\dfrac{4\sqrt{39.6} \times 3\sqrt[3]{81.2}}{21.31\sqrt{72.54}}$.
24. $\dfrac{(21.42)^{2/3} \times (1.114)^{3/2}}{(38.26)^{1/5}}$.
25. $\left(\dfrac{28.96 \times \sqrt[3]{25.05}}{\sqrt{81.7} \times 110.1}\right)^{1/2}$.
26. $\left(\dfrac{\sqrt{62.3} \times \sqrt[3]{31.24}}{76.91 \times \sqrt{0.0163}}\right)^{1/4}$.

27. Find the area of a triangle whose base and altitude are 1.683 ft and 0.9621 ft, respectively.

28. Find the area and circumference of a circle whose diameter is 2.426 in.

29. The surface S and the volume V of a sphere of radius r are given by the formulas $S = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$. Find the surface and volume of a sphere whose diameter is 2.03 in.

30. The volume V of a right circular cone of base radius r and altitude h is given by the formula $V = \frac{1}{3}\pi r^2 h$. Find the volume of a right circular cone whose base radius is 0.7561 in. and whose altitude is 4.023 in.

31. If a, b,and c are the sides of a triangle and $s = \frac{1}{2}(a + b + c)$, the area K of the triangle is given by the formula $K = \sqrt{s(s - a)(s - b)(s - c)}$. Find the area of a triangle whose sides are 5.21, 7.03, and 10.2.

32. Find the area of a triangle whose sides are 11.3, 15.2, and 21.1.

33. The period t (seconds) of a simple pendulum is given by the formula $t = 2\pi\sqrt{\dfrac{l}{g}}$, where l (feet) is the length of the pendulum and $g = 32.17$ ft per second^2 is the acceleration of gravity. Find the period of a pendulum 15 in. long.

34. Find the length of a pendulum whose period is 1 second.

In each of Exs. 35–40, find any real solutions to 4 significant figures.

35. $5^{2x} = 7^{x-1}$.

36. $e^x - e^{-x} = 2$.

37. $6^x = 3^{2x-1}$.

38. $e^x + 10e^{-x} - 7 = 0$.

39. $e^{2x} - 4e^x = 21$.

40. $e^{3x} + 4e^{2x} - 11e^x = 30$.

17

Interest and annuities

17.1. INTRODUCTION

This chapter considers briefly some of the more common financial transactions encountered by the average person. Such matters may be roughly divided into two broad classifications: (1) income from investments and (2) payments, generally of a periodic nature, for some particular future objective. Under the first item we have the income derived in the form of interest and dividends from savings accounts, stocks, and bonds. Under the second item we consider payments, usually each of a fixed amount, made at regular intervals for various purposes. Examples of such payments are those made in connection with mortgages, installment purchases, insurance policies, pension plans, and the creation of special funds.

The student will realize that in assigning only a short chapter to a few problems of finance, we are merely giving a brief introduction to a subject of considerable magnitude and importance. Entire treatises are devoted solely to the theory and applications of financial mathematics. The subject is naturally of vital concern to financial institutions, insurance companies, and business enterprises.

17.2. SIMPLE INTEREST

By *interest* on a sum of money called the *principal*, we mean a fee charged for the use of that money. The interest is a fractional part of the principal; this fraction, expressed as a per cent, is called the *rate of interest* and is generally quoted for a period of 1 year. Thus, a rate of 4% means that for each dollar loaned, the borrower must pay 4 cents as the interest for 1 year.

There are two types of interest, simple and compound. The former will be considered in this section and the latter in the next. When interest is paid at the end of a specified period of time and is computed only on the original principal, it is called *simple interest.* Generally simple interest is paid for comparatively short periods of time. For example, consider a \$1000 bond which pays interest semiannually at the rate of 4% per year. Then at the end of a specified period of 6 months, the simple interest on the bond is equal to \$1000 × 0.02 or \$20.

We will now consider the general problem of simple interest. Let P represent the principal, i the rate of interest for each of n periods of time, and I the simple interest at the end of n periods. Then

$$I = Pni. \tag{1}$$

The sum of the principal and interest is called the *amount* and is here designated by A. Hence, from (1),

$$A = P + Pni = P(1 + ni). \tag{2}$$

The *present value* of an amount A as given by (2) is *defined* as the sum of money which must be invested now at the rate i per period in order to equal A at the end of n periods of time. Evidently the present value of A is P and, from (2), is given by

$$P = A(1 + ni)^{-1}. \tag{3}$$

For convenient reference we record these results as

Theorem 1. *Let P be the principal and the present value of an amount A, I the simple interest at the end of n periods of time, and i the rate of interest for each period. Then*

$$I = Pni; \; A = P(1 + ni); \; P = A(1 + ni)^{-1}.$$

Example 1. Let the annual rate of interest be 6%. (a) Find the simple interest and amount of \$500 at the end of 3 months.

(b) Find the present value of \$600 due in 6 months at simple interest.

SOLUTION. (a) The rate i for a period of 3 months is $\dfrac{0.06}{4} = 0.015$. Hence, for $P = \$500$, the simple interest by (1) is

$$I = Pni = \$500 \times 0.015 = \$7.50,$$

and by (2), the amount is

$$A = P + Pni = \$500 + \$7.50 = \$507.50.$$

(b) The rate i for a period of 6 months is $\frac{0.06}{2} = 0.03$. By (3), the present value is

$$P = \frac{A}{1 + ni} = \frac{\$600}{1 + 0.03} = \$582.52.$$

In connection with loans at simple interest and for short periods of time, it is customary for banks to charge the interest at the time the loan is made. This deduction is called *bank discount*. Thus, for a loan of \$1000 at 6% for a period of 6 months, the bank discount is \$1000 × 0.03 = \$30. The borrower then actually receives \$1000 − \$30 or \$970 although he is required to repay the entire loan of \$1000 at the end of 6 months. Evidently the rate of interest to the borrower is then greater than 6%. This fact is also illustrated in

Example 2. A man borrows \$2000 from a bank for a period of 3 months at 5%. If he pays the bank discount, find his actual rate of interest.

SOLUTION. The bank discount is \$2000 × $\frac{1}{4}$ × 0.05 = \$25. The borrower then receives \$2000 − \$25 or \$1975, which we may regard as the present value of an amount of \$2000 due at the end of 3 months. Let r be the annual rate of interest required for \$1975 to amount to \$2000 at the end of 3 months. The interest I = \$25, so that from relation (1) above, $I = Pni$, we have

$$25 = 1975 \cdot \tfrac{1}{4}r$$

whence
$$r = \frac{100}{1975} = 5.06\%.$$

17.3. COMPOUND INTEREST

When simple interest becomes due at the end of a specified period, it may be added to the original principal to form a new principal. The interest for the next period is then computed on the new principal. If this process is carried on for two or more periods, the total increase above the original principal is called the *compound interest*. The sum of the original principal and the compound interest is called the *compound amount*. The time interval between two successive conversions of interest into principal is called the *interest period* or *conversion period*. The common interest periods are a year, 6 months, and 3 months, and the principal is then said to be compounded annually, semiannually, and quarterly, respectively. While compound interest is computed for each period at the rate for that period, the rate of interest, as in the case of simple interest, is quoted on an annual basis and is called the *nominal rate*.

As an illustration of the preceding terms, let us observe the effect of compounding on an original principal of \$1000 which is to be compounded quarterly at a nominal rate of 4%. The rate of interest for each interest period of 3 months is then 1%. We exhibit the compounding for 1 year in Table 1.

TABLE 1

Quarterly Period	Principal at Beginning of Period	Interest for Period	At End of Period	
			Comp'd Int.	Comp'd Am't.
First	1000.00	$1000 \times 0.01 = 10.00$	10.00	1010.00
Second	1010.00	$1010 \times 0.01 = 10.10$	20.10	1020.10
Third	1020.10	$1020.10 \times 0.01 = 10.20$	30.30	1030.30
Fourth	1030.30	$1030.30 \times 0.01 = 10.30$	40.60	1040.60

We will now consider the general problem of compound interest. Let

r = nominal (annual) rate of interest,
m = number of conversion periods per year,
i = interest rate for each conversion period = r/m,
n = total number of conversion periods,
P = original principal,
A_n = compound amount at end of n periods.

At the end of the first period, the interest is Pi and the amount is

(1) $$A_1 = P + Pi = P(1 + i).$$

The principal at the beginning of the second period is then A_1 and the interest at the end of that period is $A_1 i$ so that the amount at the end of the second period is

$$A_2 = A_1 + A_1 i = A_1(1 + i)$$

From (1) $$= P(1 + i)(1 + i) = P(1 + i)^2.$$

Similarly, we find that the amount at the end of the third period is

$$A_3 = P(1 + i)^3.$$

Continuing this process, we find that the compound amount at the end of n periods is

(2) $$A_n = P(1 + i)^n.$$

In relation (2), the original principal P is called the *present value* of the amount A_n. From (2) we have

(3) $$P = A_n(1 + i)^{-n}.$$

The difference $A_n - P$ is the total compound interest accumulated at the end of n periods; it is also called the *discount* on A_n.

We record the preceding results as

Theorem 2. *Let P be the original principal and the present value of the compound amount A_n at the end of n periods when i is the interest rate for each period. Then*

$$A_n = P(1 + i)^n; \ P = A_n(1 + i)^{-n}.$$

It is evident from the formulas of Theorem 2 that problems in compound interest may be computed by means of logarithms (Sec. 16.8). However, such problems may also be solved by means of tables of values of $(1 + i)^n$ and $(1 + i)^{-n}$. Short tables of this type are given in Appendix II.

Example 1. Find the compound amount at the end of 5 years for a sum of $1000 invested at a nominal rate of 6% and compounded quarterly.

SOLUTION. In this problem we are to find A_n when

$$P = \$1000, \ i = \frac{0.06}{4} = 0.015, \text{ and } n = 5 \times 4 = 20.$$

In Table 3 (Appendix II) for the compound amount $(1 + i)^n$ of $1 at the end of n periods, we find that when $i = 1\frac{1}{2}\%$ and $n = 20$, $(1 + i)^n = 1.3469$. Hence, for $P = \$1000$, we have

$$A_n = P(1 + i)^n = 1000(1 + 0.015)^{20} = 1000(1.3469) = \$1346.90.$$

The solution by logarithms is as follows:

$$\log (1 + 0.015)^{20} = 20 \log 1.015 = 20(0.0065) = 0.1300$$

Hence,

$$\log A_n = \log 1000 + \log (1.015)^{20}$$
$$= 3 + 0.1300 = 3.1300,$$

whence

$$A_n = \$1349.$$

The discrepancy in these two results is due to the fact that our table of logarithms is to four places only. A table of logarithms to six places gives a result agreeing with that obtained by the table for $(1 + i)^n$.

In computing compound interest at the end of many conversion periods, it is necessary to use logarithms of $1 + i$ to six or more places.

Example 2. Determine the present value of $4000 due at the end of 4 years if the nominal rate is 4% compounded semiannually.

SOLUTION. In this problem we are to find P when $A_n = 4000$, $i = 2\%$ and $n = 4 \times 2 = 8$.

In Table 4 (Appendix II) for the present value $(1 + i)^{-n}$ of \$1 due at the end of n periods, we find that when $i = 2\%$ and $n = 8$, $(1 + i)^{-n} = 0.85349$. Hence, for $A_n = 4000$, we have

$$P = A_n(1 + i)^{-n} = 4000(1 + 0.02)^{-8} = 4000(0.85349) = \$3413.96$$

The solution of this problem by logarithms is left as an exercise to the student.

If interest is compounded annually, the amount at the end of 1 year is the same as that obtained in simple interest. But if interest is compounded more than once during the year, the amount at the end of the year is greater than that obtained in simple interest. For example, one dollar at 6% simple interest amounts to \$1.06 at the end of the year. But if interest at the same nominal rate is compounded semiannually, one dollar amounts to $(1 + 0.03)^2 = \$1.0609$ at the end of the year. In this latter case, the interest rate for 1 year is 6.09% and is greater than the nominal rate of 6%. We then say that 6.09% is the *effective rate* of interest.

In general, the interest rate for 1 year that is equivalent to a given rate for each conversion period is called the *effective rate*. We will now find the relation between the nominal and effective rates of interest. Let r represent the nominal rate compounded n times a year, and let j represent the equivalent effective rate. Then, by definition of effective rate, we must have

$$1 + j = \left(1 + \frac{r}{n}\right)^n.$$

We record this result as

Theorem 3. *If r is the nominal rate of interest compounded n times a year and j is the equivalent effective rate of interest, then j and r are connected by the relation*

$$1 + j = \left(1 + \frac{r}{n}\right)^n.$$

Example 3. Find the effective rate of interest equivalent to a nominal rate of 5% compounded semiannually.

SOLUTION. By Theorem 3,

$$j = \left(1 + \frac{r}{n}\right)^n - 1 = (1 + 0.025)^2 - 1$$

By Table 3, $\qquad = 1.0506 - 1 = 0.0506 = 5.06\%.$

Consider now that interest is compounded n times a year. We have seen that as n increases, the effective rate of interest also increases. The student may therefore get the impression that if n is increased without limit, the effective rate is also increased without limit. This, however, is not the case, as we shall now see.

Let r be the nominal rate and let interest be compounded n times a year. Then by Theorem 2, the amount a_n of \$1 at the end of 1 year is given by

$$a_n = \left(1 + \frac{r}{n}\right)^n.$$

Now let n increase without limit, symbolically expressed as $n \to \infty$ (Sec. 10.5). We then say that interest is *compounded continuously*. It is shown in the calculus that

$$\lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^n = e^r,$$

where e is the base of the Napierian system of logarithms and is equal to $2.71828 \cdots$ (Sec. 16.4). Since e and r are both finite quantities, e^r is also a finite quantity. Hence, no matter how large a value is assigned to n, a_n is finite and, consequently, by Theorem 3, the effective rate $j = a_n - 1$ is also finite. For example, for a nominal rate $r = 6\%$, the amount of \$1 at the end of 1 year, when compounded continuously, has the limiting value $e^{0.6} = (2.71828 \cdots)^{0.6} = \$1.06184$, and the effective rate is limited to 6.184%.

To point up the effects of increasing the number of times interest is compounded during 1 year, Table 2 is given.

TABLE 2

AMOUNT AT END OF 1 YEAR FOR \$1 AT 6% ANNUAL INTEREST COMPOUNDED n TIMES PER YEAR

n	Amount
1	1.06000
2	1.06090
3	1.06121
4	1.06136
6	1.06152
12	1.06168
24	1.06176
∞	1.06184

A study of the values in this table should help to dispel many of the erroneous conceptions of the effects of the compounding of interest.

EXERCISES. GROUP 62

1. Find the simple interest on \$500 for 6 months at the annual rate of 6 %.

2. Find the simple interest on \$800 for 10 months at 4 %.

3. Find the amount of \$750 for 4 months at 5 % simple interest.

4. Find the amount of \$2100 for 8 months at $3\frac{1}{2}$ % simple interest.

5. Find the present value of \$1000 due in 3 months at 4 % simple interest.

6. Find the present value of \$1200 due in 6 months at 5 % simple interest.

7. A principal of \$1500 amounts to \$1530 at the end of 8 months. Find the rate of simple interest.

8. A loan of \$3600 is repaid by a payment of \$3654 at the end of 4 months. Find the rate of simple interest.

9. A man borrows \$3000 from a bank for a period of 6 months at 6 %. If he pays the bank discount, find his actual rate of interest.

10. A man borrows \$4000 from a bank for a period of 9 months at 5 %. If he pays the bank discount and a service charge of \$10 in advance, find his actual rate of interest.

11. Determine how long it will take a sum of money to double itself at 4 % simple interest.

12. Determine how long it will take a sum of money to double itself at 5 % simple interest.

13. Find the rate of interest if a sum of money is to double itself in 40 years at simple interest.

14. Establish relation (2) of Sec. 17.3 by mathematical induction.

15. Obtain relation (2) of Sec. 17.3 as the $(n + 1)$th term of a geometric progression whose first term is P and whose common ratio is $1 + i$ (Sec. 10.3).

16. Find the compound amount at the end of 4 years for a sum of \$500 invested at a nominal rate of 5 % and compounded semiannually. Use Table 3 (Appendix II).

17. Solve Ex. 16 by logarithms.

18. Find the compound amount at the end of 6 years for a sum of \$800 invested at a nominal rate of 8 % and compounded quarterly.

19. Find the total accumulated compound interest in Ex. 18.

20. Solve Example 2 of Sec. 17.3 by logarithms.

21. Determine the present value of \$5000 due at the end of 2 years if the nominal rate is 6 % compounded quarterly. Use Table 4 (Appendix II).

22. Solve Ex. 21 by logarithms.

23. A man invests \$2000 at 3 % compounded semiannually in order to create a special fund available 10 years hence. Find the amount of this fund.

24. A man wishes to have a fund of \$8000 available for his son's college education 16 years hence. How much must he invest now for this purpose at 4% compounded semiannually?

25. In how many years will \$2000 amount to \$5000 if invested now at 6% compounded annually? Solve by logarithms.

26. Solve Ex. 25 by linear interpolation in Table 3 (Appendix II).

27. Determine how long it will take a sum of money to double itself at 5% compounded semiannually. Solve by logarithms.

28. Solve Ex. 27 by linear interpolation in Table 3 (Appendix II).

29. Solve the relation of Theorem 3 (Sec. 17.3) for j in terms of r and also for r in terms of j.

30. Find the effective rate of interest equivalent to a nominal rate of 6% compounded quarterly.

31. Show how Table 3 (Appendix II) may be used to determine values of $(1 + i)^n$ for positive integral values of $n > 50$.

32. Show how the binomial theorem (Sec. 7.4) may be used to compute compound amount.

33. Evaluate $(1 + 0.015)^6$ by the binomial theorem and compare the result with the value given in Table 3 (Appendix II).

34. Find the nominal rate compounded semiannually equivalent to an effective rate of 4%. Use the binomial theorem.

35. Verify the figures given in Table 2 at the close of Sec. 17.3.

17.4. ANNUITIES

An *annuity* is a sequence of equal periodic payments. Simple examples of annuities are the monthly payment of rent and the payment of premiums on life insurance policies.

The term *annuity* seems to imply that payments are made annually; this, however, is not necessarily the case. The time intervals between payments may be of any length but for any particular annuity, they are considered to be equal. The time interval between two successive payments is called the *period.* In our work here it shall be understood that equal payments are made at the *end* of each period; such a sequence is called an *ordinary annuity.* The time elapsed between the beginning of the first period and the end of the last period is called the *term* of an annuity.

Consider that each payment of an annuity earns compound interest from the time the payment is made until the end of the term, each period being considered a conversion period. Then the *amount of an annuity* at the end of its term is *defined* as the sum of the compound amounts of all the payments of the annuity accumulated at the end of the term. In Table 3

TABLE 3

Quarterly Period	Payment at End of Period	Compound Amount of Payment of End of Period
First	\$100	$100(1.015)^3 =$ \$104.57
Second	100	$100(1.015)^2 =$ 103.02
Third	100	$100(1.015) =$ 101.50
Fourth	100	100.00
		Amount of Annuity = \$409.09

we illustrate the various steps in obtaining the amount of an annuity for a term of 1 year, where payments of \$100 each are made at the end of each of four periods, the nominal rate of interest being 6%. The interest rate for each period of 3 months is then 1.5%.

It is evident that the amount of an annuity is obtained as the sum of a geometric progression. This is also seen in the following determination of the general formula for the amount of an ordinary annuity.

Let us now consider an ordinary annuity where R is the payment made at the end of each of n periods and i is the rate of interest for each period. Since the first payment is made at the end of the first period, it will accumulate interest for $n - 1$ periods and, by Theorem 2 (Sec. 17.3), will amount to $R(1 + i)^{n-1}$ at the end of the term. Similarly, the second payment will accumulate interest for $n - 2$ periods and will amount to $R(1 + i)^{n-2}$ at the end of the term. Continuing this way, we see that the $(n - 1)$th payment will amount to $R(1 + i)$ and that the nth or final payment amounts to simply R. Writing these amounts in reverse order, we have

$$R,\ R(1 + i),\ R(1 + i)^2,\ \cdots,\ R(1 + i)^{n-2},\ R(1 + i)^{n-1},$$

which is a geometric progression of n terms with $1 + i$ as its common ratio. By definition, the sum S of this progression is the amount of the annuity and, by Theorem 2 of Sec. 10.3, its value is given by

(1) $$S = R\,\frac{1 - (1 + i)^n}{1 - (1 + i)} = R\,\frac{(1 + i)^n - 1}{i}.$$

For the particular case where $R = 1$, S is designated by the symbol $s_{\overline{n}|i}$ so that from (1),

(2) $$s_{\overline{n}|i} = \frac{(1 + i)^n - 1}{i}$$

and

(3) $$S = Rs_{\overline{n}|i}.$$

To facilitate the solution of problems in annuities, tables of values of $s_{\overline{n}|i}$ have been made up. A short table of this type is given as Table 5 in Appendix II.

We next consider the *present value of an annuity* which is *defined* as the sum of the present values of all the payments. As before, for an ordinary annuity, let R be the payment made at the end of each of n periods and let i be the rate of interest for each period. Then by Theorem 2 (Sec. 17.3), the present value of the first payment (made at the end of the first period) is $R(1 + i)^{-1}$; the present value of the second payment (made at the end of the second period) is $R(1 + i)^{-2}$; and so on. The present value of the last or nth payment is $R(1 + i)^{-n}$. The present value A of the annuity is the sum of these present values of the payments, and hence

$$A = R(1 + i)^{-1} + R(1 + i)^{-2} + \cdots + R(1 + i)^{-n},$$

a geometric progression of n terms with $(1 + i)^{-1}$ as its common ratio. Then, by Theorem 2 of Sec. 10.3,

$$A = \frac{R(1 + i)^{-1}[1 - (1 + i)^{-n}]}{1 - (1 + i)^{-1}}.$$

Multiplying numerator and denominator by $1 + i$, we obtain

$$(4) \qquad A = \frac{R[1 - (1 + i)^{-n}]}{1 + i - 1} = R\frac{1 - (1 + i)^{-n}}{i}.$$

For the particular case where $R = 1$, A is designated by the symbol $a_{\overline{n}|i}$ so that from (4),

$$(5) \qquad a_{\overline{n}|i} = \frac{1 - (1 + i)^{-n}}{i}$$

and

$$(6) \qquad A = Ra_{\overline{n}|i}.$$

A short table of values of $a_{\overline{n}|i}$ is given as Table 6 in Appendix II.

For convenient reference we record the preceding results as

Theorem 4. *Let S be the amount and A the present value of an ordinary annuity consisting of n payments of R with i the rate of interest for each period. Then*

$$S = R\frac{(1 + i)^n - 1}{i}; \quad A = R\frac{1 - (1 + i)^{-n}}{i}.$$

Example. An ordinary annuity consists of payments of \$300 made semiannually for 5 years, the nominal rate of interest being 3%. Find (a) the amount and (b) the present value of this annuity.

SOLUTION. (a) For this annuity, $R = 300$, the number of periods $n = 10$, and the interest rate for each period $i = 1.5\%$. For $n = 10$ and $i = 1.5\%$, Table 5 (Appendix II) gives the value $s_{\overline{n}|i} = 10.7027$. Hence, by relation (3), the required amount is

$$S = Rs_{\overline{n}|i} = 300(10.7027) = \$3210.81.$$

(b) For $n = 10$ and $i = 1.5\%$, Table 6 (Appendix II) gives the value $a_{\overline{n}|i} = 9.2222$. Hence, by relation (6), the required present value is

$$A = Ra_{\overline{n}|i} = 300(9.2222) = \$2766.66.$$

NOTE. Annuity computations may also be made by means of logarithms or the binomial theorem. For great accuracy, particularly for annuities over long terms, extensive tables of logarithms and tables of $s_{\overline{n}|i}$ and $a_{\overline{n}|i}$ are required.

17.5. APPLICATIONS OF ANNUITIES

In this section we discuss several examples of some financial transactions which are essentially problems in annuities.

Sinking Fund

A sum of money accumulated to pay an obligation due at some future date is called a *sinking fund*. Such a fund does not include any payments of interest on the obligation; these payments are considered as a separate item. A sinking fund is usually created by investing equal sums of money at the end of equal periods of time. It is therefore the amount S of an annuity as given in Theorem 4 (Sec. 17.4).

Sinking funds are commonly established for retiring a loan due at a future date, for example, the redemption of a bond issue. However, sinking funds may be created for other purposes. Thus, a sinking fund is often used for replacing worn-out or obsolete equipment; it is then called a *depreciation fund*. Note that a depreciation fund does not include any allowance for current maintenance charges and interest on invested capital.

Example 1. The useful life of some manufacturing equipment is 10 years and the net cost of replacing it at the end of that time is \$15,000. Determine the semiannual amounts which must be invested at a nominal rate of 5% to create the required depreciation fund.

SOLUTION. In this problem we are required to find the periodic payment R of an annuity whose amount S is \$15,000 at the end of $n = 20$ periods and whose rate of interest for each period is $i = 2.5\%$.

For $n = 20$ and $i = 2.5\%$, Table 5 (Appendix II) gives the value $s_{\overline{n}|i} = 25.5447$.

From relation (3), Sec. 17.4, $S = Rs_{\overline{n}|i}$. Hence

$$R = \frac{S}{s_{\overline{n}|i}} = \frac{15000}{25.5447} = \$587.20 \text{ (very nearly).}$$

Amortization

The discharge of an interest-bearing debt by equal payments made at the end of equal periods of time is called *amortization*. It is clear that each payment must exceed the interest charge on the debt for the first period. These payments serve to decrease the debt each period. As a result, the part of each payment that is used to pay interest on the debt is decreasing and the rest of the payment applied to the debt is increasing correspondingly. This shift in the distribution of each payment to debt and interest is illustrated in the amortization schedule accompanying Example 2 below. The student should note that amortization of a debt differs from the creation of a sinking fund by the fact that it provides not only for the payment of the debt but also for the payment of the intervening interest on the debt.

It is evident from the definition above that amortization of a debt is effected by means of an annuity. The debt to be amortized is the present value A of an annuity. If R is the payment at the end of each of n periods and i is the interest rate for each period, then the debt A to be amortized is given by relation (4) of Sec. 17.4, namely, $A = Ra_{\overline{n}|i}$.

Example 2. A loan of \$4000 is to be amortized by five equal annual payments. Find the annual payment if the interest rate is 5% compounded annually.

SOLUTION. In this problem we are required to find the annual payment R of an annuity whose present value $A = \$4000$, whose term is $n = 5$ periods, and whose rate of interest i for each period is 5%.

For $n = 5$ and $i = 5\%$, Table 6 (Appendix II) gives the value $a_{\overline{n}|i} = 4.3295$. Hence from the relation $A = Ra_{\overline{n}|i}$, we have

$$R = \frac{A}{a_{\overline{n}|i}} = \frac{4000}{4.3295} = \$923.90.$$

It is instructive to observe, for each of the five annual periods, the distribution of each annual payment between interest and principal (debt). This is shown in Table 4, which is appropriately called an *amortization schedule*.

The figures in column (3) are 5% of the corresponding figures in column

TABLE 4

	(1)	(2)	(3)	(4)
Year	Principal at Beginning of Year	Annual Payment at End of Year	Interest Paid at End of Year	Principal Repaid at End of Year
1	\$4000.00	\$923.90	\$200.00	\$723.90
2	3276.10	923.90	163.81	760.09
3	2516.01	923.90	125.80	798.10
4	1717.91	923.90	85.90	838.00
5	879.91	923.90	43.99	879.91
	Totals	4619.50	619.50	4000.00

(1). The figures in column (4) are obtained by subtracting the corresponding figures in column (3) from the annual payment of \$923.90 given in column (2).

One of the most common examples of amortization is the payment of a mortgage on a house. This payment is usually made by means of equal monthly installments, each being greater than the first month's interest on the mortgage. The lending institution generally supplies the borrower with an amortization schedule showing the distribution of each monthly installment between interest and principal.

We will now derive a useful formula which gives the time required to amortize a mortgage. For this purpose let

$R =$ the monthly installment,
$i =$ the monthly rate of interest,
$n =$ number of months to amortize mortgage,
$M =$ the original mortgage.

From Theorem 4 (Sec. 17.4), with $A = M$, the formula for the present value of an annuity is

$$M = R\frac{1-(1+i)^{-n}}{i}$$

whence

$$\frac{Mi}{R} = 1-(1+i)^{-n}$$

and

$$(1+i)^{-n} = 1 - \frac{Mi}{R} = \frac{R-Mi}{R}.$$

Then

$$(1+i)^{n} = \frac{R}{R-Mi}$$

and

$$n \log (1+i) = \log R - \log (R-Mi),$$

from which we have our required formula

$$(1) \qquad n = \frac{\log R - \log (R - Mi)}{\log (1 + i)}.$$

Example 3. Find the number of months required to amortize a mortgage of \$1000 by equal monthly payments of \$10, interest being at the annual rate of 6%.

SOLUTION. Here, $R = 10$, $i = 0.005$, and $M = 1000$. Substituting these values in relation (1) above, we obtain

$$n = \frac{\log 10 - \log (10 - 5)}{\log (1 + 0.005)} = 139 \text{ months (very nearly).}$$

EXERCISES. GROUP 63

1. In Theorem 4 of Sec. 17.4, show that $A = S(1 + i)^{-n}$. Verify this relation for the example of Sec. 17.4.

2. By means of logarithms, obtain the value of the amount S in the example of Sec. 17.4.

3. By means of the binomial theorem, obtain the value of the amount S in the example of Sec. 17.4.

In each of Exs. 4–7, find the amount and present value of the annuity described.

4. Payments of \$200 quarterly for 4 years, the nominal rate of interest being 6%.

5. Payments of \$500 annually for 10 years, the nominal rate of interest being 4%.

6. Payments of \$400 semiannually for 12 years, the nominal rate of interest being 5%.

7. Payments of \$300 annually for 6 years, the nominal rate of interest being 3%.

8. Find the amount of each payment which must be made quarterly to accumulate an amount of \$2000 at the end of 5 years, the nominal rate of interest being 6%.

9. Find the present value of the annuity described in Ex. 8.

10. If an annuity continues for an unlimited time, it is said to become a *perpetuity*, and we indicate this fact symbolically by writing $n \to \infty$. Using relation (4) of Sec. 17.4, show that the present value of a perpetuity is equal to Ri^{-1}.

11. Using the result of Ex. 10, find the present value of a perpetuity paying \$1000 semiannually, the nominal rate of interest being 4%.

12. A perpetuity whose present value is \$10,000 pays \$125 quarterly. Find the nominal rate of interest.

13. A man wishes to accumulate a fund of \$8000 to be available 15 years hence for his son's college education. Determine how much he must deposit in a bank at the end of each period of 6 months, if interest is compounded semiannually, the nominal rate being 4%.

14. Find the number of quarterly payments of \$100 each that must be made to accumulate a sum of \$6000, the nominal rate of interest being 6%.

15. Find the number of semiannual payments of \$300 each that must be made to accumulate a sum whose present value is \$2700, the nominal rate of interest being 5%.

16. A man owns a bond which is due in 10 years and pays semiannual dividends of \$20 each. As each dividend is received, it is invested at a nominal rate of 4% compounded semiannually. Find the amount of this investment at the time the bond is redeemed.

17. A bond whose redemption price at the end of 10 years is \$2000 has 20 coupons attached, each coupon having a value of \$40 semiannually. Find the present value of both these coupons and the bond on the basis of a nominal rate of 5% compounded semiannually.

18. In order to construct a school, a town obtains \$200,000 by means of a bond issue due in 15 years. Determine the amount to be deposited semiannually in a sinking fund to redeem this issue if the return on these deposits is at the nominal rate of 4% compounded semiannually.

19. A company finds that the useful life of a truck is 8 years at the end of which time the replacement cost is \$5000. Find the amount which must be invested quarterly at a nominal rate of 6% in order to accumulate the replacement cost.

20. A man agrees to repay a debt of \$6000 in one lump sum at the end of 5 years. If he uses the sinking fund method for this purpose, determine how much he must invest semiannually at a nominal rate of 5% compounded semiannually.

21. If in Ex. 20, the borrower must also pay interest semiannually at a nominal rate of 4%, find the total semiannual expense of his debt.

22. A man wishes to amortize a debt of \$10,000 at the end of 4 years by equal annual payments. Find the annual payment if the interest rate is 4% compounded annually.

23. Construct an amortization schedule for Ex. 22.

24. A company borrows \$50,000 to modernize its plant. To amortize this debt, it makes equal quarterly payments for a period of 2 years. Find the amount of each payment if the nominal rate is 6% compounded quarterly.

25. Construct an amortization schedule for Ex. 24.

26. A man wishes to provide an annual scholarship of \$1000 for each of 10 years. How much must he invest for this purpose at the beginning of the 10-year period at 5% compounded annually?

27. Verify the result of Example 3 of Sec. 17.5.

28. How long will it take to amortize a mortgage of \$8000 by equal monthly payments of \$50, interest being at the nominal rate of $4\frac{1}{2}\%$?

29. A man wishes to amortize a mortgage of \$10,000 by equal monthly payments over a period of 10 years. Find the amount of each payment if the nominal rate of interest is 6%.

30. A man borrows \$1200 from a bank at the nominal rate of $4\frac{1}{2}$%, with the agreement that he is to pay the bank discount for 1 year at once and pay \$100 per month for 12 months. Determine his actual rate of interest.

Appendix I

Reference lists and data

A. Bibliography

A short list of references is given. These treatises cover many topics in algebra which are beyond the scope of this book. The list must be considered suggestive and not at all inclusive; there are numerous other books which are both helpful and informative.

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B. Trigonometry

1. Definitions of the trigonometric functions. Let θ be an angle whose range of values is given by $-360° \leq \theta \leq 360°$. For the purposes of defining such an angle and its trigonometric functions it is convenient to use the rectangular coordinate system. The statements which follow apply to *each* of the four positions shown in Fig. 45.

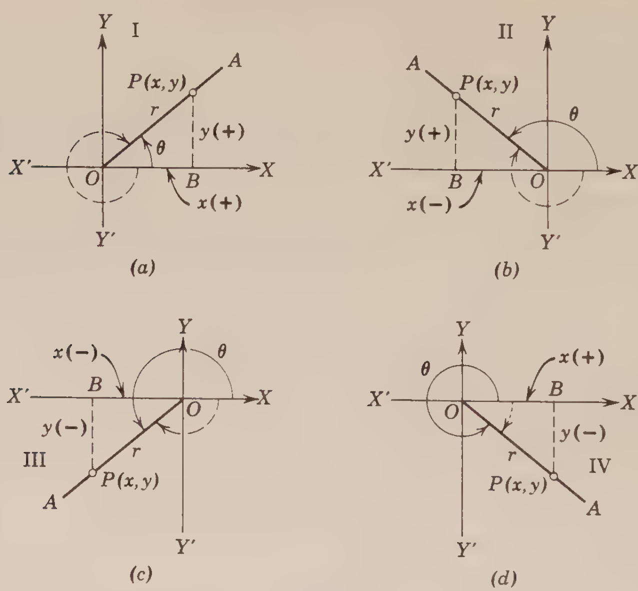

Figure 45

If a line coincident with the X-axis is rotated in the XY-coordinate plane about the origin O into a new position OA, there is said to be *generated* an angle $XOA = \theta$ having OX as its *initial side* and OA as its *terminal side.* If the rotation is counterclockwise, the angle generated is said to be *positive;* for clockwise rotation (shown dotted in the figures), the angle is said to be *negative.* The angle is said to lie in the same quadrant as its terminal side.

On the terminal side OA take any point P distinct from O and having coordinates (x, y). From P drop a perpendicular PB to the X-axis. The line segment OP is called the *radius vector*, is designated by r, and is always taken as *positive*. In the triangle OPB, $OB = x$ and $PB = y$ have the signs of the coordinates of the point P, as indicated for the four quadrants. Then, irrespective of the quadrant in which θ lies, the six trigonometric functions of θ are *defined* both as to magnitude and sign by the following ratios:

$$\text{sine of } \theta\text{: } \sin\theta = \frac{y}{r}, \qquad \text{cosine of } \theta\text{: } \cos\theta = \frac{x}{r},$$

$$\text{tangent of } \theta\text{: } \tan\theta = \frac{y}{x}, \qquad \text{cotangent of } \theta\text{: } \cot\theta = \frac{x}{y},$$

$$\text{secant of } \theta\text{: } \sec\theta = \frac{r}{x}, \qquad \text{cosecant of } \theta\text{: } \csc\theta = \frac{r}{y}.$$

The definitions hold without change for positive and negative angles greater than 360° in numerical value.

2. Fundamental trigonometric identities

$$\csc\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta}, \quad \tan\theta = \frac{\sin\theta}{\cos\theta},$$

$$\sin^2\theta + \cos^2\theta = 1, \quad 1 + \tan^2\theta = \sec^2\theta, \quad 1 + \cot^2\theta = \csc^2\theta.$$

3. Reduction formulas

$$\sin(90° \pm \theta) = \cos\theta, \quad \cos(90° \pm \theta) = \mp\sin\theta, \quad \tan(90° \pm \theta) = \mp\cot\theta,$$

$$\sin(180° \pm \theta) = \mp\sin\theta, \quad \cos(180° \pm \theta) = -\cos\theta, \quad \tan(180° \pm \theta) = \pm\tan\theta,$$

$$\sin(270° \pm \theta) = -\cos\theta, \quad \cos(270° \pm \theta) = \pm\sin\theta, \quad \tan(270° \pm \theta) = \mp\cot\theta,$$

$$\sin(360° \pm \theta) = \pm\sin\theta, \quad \cos(360° \pm \theta) = \cos\theta, \quad \tan(360° \pm \theta) = \pm\tan\theta.$$

4. Radian measure of angles. Let θ be a central angle intercepting an arc of length s on a circle of radius r. Then the measure of the angle θ in *radians* is *defined* by $\theta = \frac{s}{r}$. Note that, since s and r are lengths, this ratio is a *pure number*. From this definition of radian measure we have at once the conversion relation:

$$\pi \text{ radians} = 180°$$

whence

$$1 \text{ radian} = \frac{180°}{\pi} = 57.2958° \text{ (approx.)} = 57°17'45'' \text{ (approx.)},$$

$$1° = \frac{\pi}{180} \text{ radian} = 0.017453 \text{ radian (approx.)}.$$

5. TRIGONOMETRIC FUNCTIONS OF SPECIAL ANGLES

Angle θ in		$\sin\theta$	$\cos\theta$	$\tan\theta$
Radians	Degrees			
0	$0°$	0	1	0
$\frac{\pi}{6}$	$30°$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$
$\frac{\pi}{4}$	$45°$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	1
$\frac{\pi}{3}$	$60°$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	$90°$	1	0	

6. ADDITION AND SUBTRACTION FORMULAS

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}.$$

7. DOUBLE-ANGLE FORMULAS

$$\sin 2x = 2 \sin x \cos x,$$
$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1,$$
$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}.$$

8. HALF-ANGLE FORMULAS

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{2}}, \quad \cos\frac{x}{2} = \pm\sqrt{\frac{1 + \cos x}{2}},$$
$$\tan\frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{1 + \cos x}} = \frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x}.$$

C. The Greek Alphabet

A	α	alpha	I	ι	iota	P	ρ	rho
B	β	beta	K	κ	kappa	Σ	σ	sigma
Γ	γ	gamma	Λ	λ	lambda	T	τ	tau
Δ	δ	delta	M	μ	mu	Υ	υ	upsilon
E	ϵ	epsilon	N	ν	nu	Φ	ϕ	phi
Z	ζ	zeta	Ξ	ξ	xi	X	χ	chi
H	η	eta	O	o	omicron	Ψ	ψ	psi
Θ	θ	theta	Π	π	pi	Ω	ω	omega

Appendix II

Tables

1. NATURAL TRIGONOMETRIC FUNCTIONS

Radians	Degrees	Sine	Cosine	Tangent	Cotangent		
.0000	0.0	.0000	1.0000	.0000	——	90.0	1.5708
.0087	0.5	.0087	1.0000	.0087	114.5887	89.5	1.5621
.0175	1.0	.0175	.9998	.0175	57.2900	89.0	1.5533
.0262	1.5	.0262	.9997	.0262	38.1885	88.5	1.5446
.0349	2.0	.0349	.9994	.0349	28.6363	88.0	1.5359
.0436	2.5	.0436	.9990	.0437	22.9038	87.5	1.5272
.0524	3.0	.0523	.9986	.0524	19.0811	87.0	1.5184
.0611	3.5	.0610	.9981	.0612	16.3499	86.5	1.5097
.0698	4.0	.0698	.9976	.0699	14.3007	86.0	1.5010
.0785	4.5	.0785	.9969	.0787	12.7062	85.5	1.4923
.0873	5.0	.0872	.9962	.0875	11.4301	85.0	1.4835
.0960	5.5	.0958	.9954	.0963	10.3854	84.5	1.4748
.1047	6.0	.1045	.9945	.1051	9.5144	84.0	1.4661
.1134	6.5	.1132	.9936	.1139	8.7769	83.5	1.4574
.1222	7.0	.1219	.9925	.1228	8.1443	83.0	1.4486
.1309	7.5	.1305	.9914	.1317	7.5958	82.5	1.4399
.1396	8.0	.1392	.9903	.1405	7.1154	82.0	1.4312
.1484	8.5	.1478	.9890	.1495	6.6912	81.5	1.4224
.1571	9.0	.1564	.9877	.1584	6.3138	81.0	1.4137
.1658	9.5	.1650	.9863	.1673	5.9758	80.5	1.4050
.1745	10.0	.1736	.9848	.1763	5.6713	80.0	1.3963
.1833	10.5	.1822	.9835	.1853	5.3955	79.5	1.3875
.1920	11.0	.1908	.9816	.1944	5.1446	79.0	1.3788
.2007	11.5	.1994	.9799	.2035	4.9152	78.5	1.3701
.2094	12.0	.2079	.9781	.2126	4.7046	78.0	1.3614
.2182	12.5	.2164	.9763	.2217	4.5107	77.5	1.3526
.2269	13.0	.2250	.9744	.2309	4.3315	77.0	1.3439
.2356	13.5	.2334	.9724	.2401	4.1653	76.5	1.3352
.2443	14.0	.2419	.9703	.2493	4.0108	76.0	1.3265
.2531	14.5	.2504	.9681	.2586	3.8667	75.5	1.3177
.2618	15.0	.2588	.9659	.2679	3.7321	75.0	1.3090
.2705	15.5	.2672	.9636	.2773	3.6059	74.5	1.3003
.2793	16.0	.2756	.9613	.2867	3.4874	74.0	1.2915
.2880	16.5	.2840	.9588	.2962	3.3759	73.5	1.2828
.2967	17.0	.2924	.9563	.3057	3.2709	73.0	1.2741
.3054	17.5	.3007	.9537	.3153	3.1716	72.5	1.2654
.3142	18.0	.3090	.9511	.3249	3.0777	72.0	1.2566
.3229	18.5	.3173	.9483	.3346	2.9887	71.5	1.2479
.3316	19.0	.3256	.9455	.3443	2.9042	71.0	1.2392
.3403	19.5	.3338	.9426	.3541	2.8239	70.5	1.2305
.3491	20.0	.3420	.9397	.3640	2.7475	70.0	1.2217
.3578	20.5	.3502	.9367	.3739	2.6746	69.5	1.2130
.3665	21.0	.3584	.9336	.3839	2.6051	69.0	1.2043
.3752	21.5	.3665	.9304	.3939	2.5386	68.5	1.1956
.3840	22.0	.3746	.9272	.4040	2.4751	68.0	1.1868
.3927	22.5	.3827	.9239	.4142	2.4142	67.5	1.1781
		Cosine	Sine	Cotangent	Tangent	Degrees	Radians

1. NATURAL TRIGONOMETRIC FUNCTIONS

Radians	Degrees	Sine	Cosine	Tangent	Cotangent		
.3927	22.5	.3827	.9239	.4142	2.4142	67.5	1.1781
.4014	23.0	.3907	.9205	.4245	2.3559	67.0	1.1694
.4102	23.5	.3987	.9171	.4348	2.2998	66.5	1.1606
.4189	24.0	.4067	.9135	.4452	2.2460	66.0	1.1519
.4276	24.5	.4147	.9100	.4557	2.1943	65.5	1.1432
.4363	25.0	.4226	.9063	.4663	2.1445	65.0	1.1345
.4451	25.5	.4305	.9026	.4770	2.0965	64.5	1.1257
.4538	26.0	.4384	.8988	.4877	2.0503	64.0	1.1170
.4625	26.5	.4462	.8949	.4986	2.0057	63.5	1.1083
.4712	27.0	.4540	.8910	.5095	1.9626	63.0	1.0996
.4800	27.5	.4617	.8870	.5206	1.9210	62.5	1.0908
.4887	28.0	.4695	.8829	.5317	1.8807	62.0	1.0821
.4974	28.5	.4772	.8788	.5430	1.8418	61.5	1.0734
.5061	29.0	.4848	.8746	.5543	1.8040	61.0	1.0647
.5149	29.5	.4924	.8704	.5658	1.7675	60.5	1.0559
.5236	30.0	.5000	.8660	.5774	1.7321	60.0	1.0472
.5323	30.5	.5075	.8616	.5890	1.6977	59.5	1.0385
.5411	31.0	.5150	.8572	.6009	1.6643	59.0	1.0297
.5498	31.5	.5225	.8526	.6128	1.6319	58.5	1.0210
.5585	32.0	.5299	.8480	.6249	1.6003	58.0	1.0123
.5672	32.5	.5373	.8434	.6371	1.5697	57.5	1.0036
.5760	33.0	.5446	.8387	.6494	1.5399	57.0	.9948
.5847	33.5	.5519	.8339	.6619	1.5108	56.5	.9861
.5934	34.0	.5592	.8290	.6745	1.4826	56.0	.9774
.6021	34.5	.5664	.8241	.6873	1.4550	55.5	.9687
.6109	35.0	.5736	.8192	.7002	1.4281	55.0	.9599
.6196	35.5	.5807	.8141	.7133	1.4019	54.5	.9512
.6283	36.0	.5878	.8090	.7265	1.3764	54.0	.9425
.6370	36.5	.5948	.8039	.7400	1.3514	53.5	.9338
.6458	37.0	.6018	.7986	.7536	1.3270	53.0	.9250
.6545	37.5	.6088	.7934	.7673	1.3032	52.5	.9163
.6632	38.0	.6157	.7880	.7813	1.2799	52.0	.9076
.6720	38.5	.6225	.7826	.7954	1.2572	51.5	.8988
.6807	39.0	.6293	.7771	.8098	1.2349	51.0	.8901
.6894	39.5	.6361	.7716	.8243	1.2131	50.5	.8814
.6981	40.0	.6428	.7660	.8391	1.1918	50.0	.8727
.7069	40.5	.6494	.7604	.8541	1.1708	49.5	.8639
.7156	41.0	.6561	.7547	.8693	1.1504	49.0	.8552
.7243	41.5	.6626	.7490	.8847	1.1303	48.5	.8465
.7330	42.0	.6691	.7431	.9004	1.1106	48.0	.8378
.7418	42.5	.6756	.7373	.9163	1.0913	47.5	.8290
.7505	43.0	.6820	.7314	.9325	1.0724	47.0	.8203
.7592	43.5	.6884	.7254	.9490	1.0538	46.5	.8116
.7679	44.0	.6947	.7193	.9657	1.0355	46.0	.8029
.7767	44.5	.7009	.7133	.9827	1.0176	45.5	.7941
.7854	45.0	.7071	.7071	1.0000	1.0000	45.0	.7854
		Cosine	Sine	Cotangent	Tangent	Degrees	Radians

2. COMMON LOGARITHMS

	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

2. COMMON LOGARITHMS

	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

3. COMPOUND AMOUNT: $(1+i)^n$

n	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	4%	5%	6%
1	1.0150	1.0200	1.0250	1.0300	1.0400	1.0500	1.0600
2	1.0302	1.0404	1.0506	1.0609	1.0816	1.1025	1.1236
3	1.0457	1.0612	1.0769	1.0927	1.1249	1.1576	1.1910
4	1.0614	1.0824	1.1038	1.1255	1.1699	1.2155	1.2625
5	1.0773	1.1041	1.1314	1.1593	1.2167	1.2763	1.3382
6	1.0934	1.1262	1.1597	1.1941	1.2653	1.3401	1.4185
7	1.1098	1.1487	1.1887	1.2299	1.3159	1.4071	1.5036
8	1.1265	1.1717	1.2184	1.2668	1.3686	1.4775	1.5938
9	1.1434	1.1951	1.2489	1.3048	1.4233	1.5513	1.6895
10	1.1605	1.2190	1.2801	1.3439	1.4802	1.6289	1.7908
11	1.1779	1.2434	1.3121	1.3842	1.5395	1.7103	1.8983
12	1.1956	1.2682	1.3449	1.4258	1.6010	1.7959	2.0122
13	1.2136	1.2936	1.3785	1.4685	1.6651	1.8856	2.1329
14	1.2318	1.3195	1.4130	1.5126	1.7317	1.9799	2.2609
15	1.2502	1.3459	1.4483	1.5580	1.8009	2.0789	2.3966
16	1.2690	1.3728	1.4845	1.6047	1.8730	2.1829	2.5404
17	1.2880	1.4002	1.5216	1.6528	1.9479	2.2920	2.6928
18	1.3073	1.4282	1.5597	1.7024	2.0258	2.4066	2.8543
19	1.3270	1.4568	1.5987	1.7535	2.1068	2.5270	3.0256
20	1.3469	1.4859	1.6386	1.8061	2.1911	2.6533	3.2071
21	1.3671	1.5157	1.6796	1.8603	2.2788	2.7860	3.3996
22	1.3876	1.5460	1.7216	1.9161	2.3699	2.9253	3.6035
23	1.4084	1.5769	1.7646	1.9736	2.4647	3.0715	3.8197
24	1.4295	1.6084	1.8087	2.0328	2.5633	3.2251	4.0489
25	1.4509	1.6406	1.8539	2.0938	2.6658	3.3864	4.2919
26	1.4727	1.6734	1.9003	2.1566	2.7725	3.5557	4.5494
27	1.4948	1.7069	1.9478	2.2213	2.8834	3.7335	4.8223
28	1.5172	1.7410	1.9965	2.2879	2.9987	3.9201	5.1117
29	1.5400	1.7758	2.0464	2.3566	3.1187	4.1161	5.4184
30	1.5631	1.8114	2.0976	2.4273	3.2434	4.3219	5.7435
31	1.5865	1.8476	2.1500	2.5001	3.3731	4.5380	6.0881
32	1.6103	1.8845	2.2038	2.5751	3.5081	4.7649	6.4534
33	1.6345	1.9222	2.2589	2.6523	3.6484	5.0032	6.8406
34	1.6590	1.9607	2.3153	2.7319	3.7943	5.2533	7.2510
35	1.6839	1.9999	2.3732	2.8139	3.9461	5.5160	7.6861
36	1.7091	2.0399	2.4325	2.8983	4.1039	5.7918	8.1473
37	1.7348	2.0807	2.4933	2.9852	4.2681	6.0814	8.6361
38	1.7608	2.1223	2.5557	3.0748	4.4388	6.3855	9.1543
39	1.7872	2.1647	2.6196	3.1670	4.6164	6.7048	9.7035
40	1.8140	2.2080	2.6851	3.2620	4.8010	7.0400	10.2857
41	1.8412	2.2522	2.7522	3.3599	4.9931	7.3920	10.9029
42	1.8688	2.2972	2.8210	3.4607	5.1928	7.7616	11.5570
43	1.8969	2.3432	2.8915	3.5645	5.4005	8.1497	12.2505
44	1.9253	2.3901	2.9638	3.6715	5.6165	8.5572	12.9855
45	1.9542	2.4379	3.0379	3.7816	5.8412	8.9850	13.7646
46	1.9835	2.4866	3.1139	3.8950	6.0748	9.4343	14.5905
47	2.0133	2.5363	3.1917	4.0119	6.3178	9.9060	15.4659
48	2.0435	2.5871	3.2715	4.1323	6.5705	10.4013	16.3939
49	2.0741	2.6388	3.3533	4.2562	6.8333	10.9213	17.3775
50	2.1052	2.6916	3.4371	4.3839	7.1067	11.4674	18.4202

4. PRESENT VALUE: $(1 + i)^{-n}$

n	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	4%	5%	6%
1	.985 22	.980 39	.97561	.97087	.96154	.95238	.94340
2	.970 66	.961 17	.95181	.94260	.92456	.90703	.89000
3	.956 32	.942 32	.92860	.91514	.88900	.86384	.83962
4	.942 18	.923 85	.90595	.88849	.85480	.82270	.79209
5	.928 26	.905 73	.88385	.86261	.82193	.78353	.74726
6	.914 54	.887 97	.86230	.83748	.79031	.74622	.70496
7	.901 03	.870 56	.84127	.81309	.75992	.71068	.66506
8	.887 71	.853 49	.82075	.78941	.73069	.67684	.62741
9	.874 59	.836 76	.80073	.76642	.70259	.64461	.59190
10	.861 67	.820 35	.78120	.74409	.67556	.61391	.55839
11	.848 93	.804 26	.76214	.72242	.64958	.58468	.52679
12	.836 39	.788 49	.74356	.70138	.62460	.55684	.49697
13	.824 03	.773 03	.72542	.68095	.60057	.53032	.46884
14	.811 85	.757 88	.70773	.66112	.57748	.50507	.44230
15	.799 85	.743 01	.69047	.64186	.55526	.48102	.41727
16	.788 03	.728 45	.67362	.62317	.53391	.45811	.39365
17	.776 39	.714 16	.65720	.60502	.51337	.43630	.37136
18	.764 91	.700 16	.64117	.58739	.49363	.41552	.35034
19	.753 61	.686 43	.62553	.57029	.47464	.39573	.33051
20	.742 47	.672 97	.61027	.55368	.45639	.37689	.31180
21	.731 50	.659 78	.59539	.53755	.43883	.35894	.29416
22	.720 69	.646 84	.58086	.52189	.42196	.34185	.27751
23	.710 04	.634 16	.56670	.50669	.40573	.32557	.26180
24	.699 54	.621 72	.55288	.49193	.39012	.31007	.24698
25	.689 21	.609 53	.53939	.47761	.37512	.29530	.23300
26	.679 02	.597 58	.52623	.46369	.36065	.28124	.21981
27	.668 99	.585 86	.51340	.45019	.34682	.26785	.20737
28	.659 10	.574 37	.50088	.43708	.33348	.25509	.19563
29	.649 36	.563 11	.48866	.42435	.32069	.24295	.18456
30	.639 76	.552 07	.47674	.41199	.30832	.23138	.17411
31	.630 31	.541 25	.46511	.39999	.29646	.22036	.16425
32	.620 99	.530 63	.45377	.38834	.28506	.20987	.15496
33	.611 82	.520 23	.44270	.37703	.27409	.19987	.14619
34	.602 77	.510 03	.43191	.36604	.26355	.19035	.13791
35	.593 87	.500 03	.42137	.35538	.25342	.18129	.13011
36	.585 09	.490 22	.41109	.34503	.24367	.17266	.12274
37	.576 44	.480 61	.40107	.33498	.23430	.16444	.11579
38	.567 92	.471 19	.39128	.32523	.22529	.15661	.10924
39	.559 53	.461 95	.38174	.31575	.21662	.14915	.10306
40	.551 26	.452 89	.37243	.30656	.20829	.14205	.09722
41	.543 12	.444 01	.36335	.29763	.20028	.13528	.09172
42	.535 09	.435 30	.35448	.28896	.19257	.12884	.08653
43	.527 18	.426 77	.34584	.28054	.18517	.12270	.08163
44	.519 39	.418 40	.33740	.27237	.17805	.11686	.07701
45	.511 71	.410 20	.32917	.26444	.17120	.11130	.07265
46	.504 15	.402 15	.32115	.25674	.16461	.10600	.06854
47	.496 70	.394 27	.31331	.24926	.15828	.10095	.06466
48	.489 36	.386 54	.30567	.24200	.15219	.09614	.06100
49	.482 13	.378 96	.29822	.23495	.14634	.09156	.05755
50	.475 00	.371 53	.29094	.22811	.14071	.08720	.05429

5. AMOUNT OF AN ANNUITY: $s_{\overline{n}|i}$

n	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	4%	5%	6%
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	2.0150	2.0200	2.0250	2.0300	2.0400	2.0500	2.0600
3	3.0452	3.0604	3.0756	3.0909	3.1216	3.1525	3.1836
4	4.0909	4.1216	4.1525	4.1836	4.2465	4.3101	4.3746
5	5.1523	5.2040	5.2563	5.3091	5.4163	5.5256	5.6371
6	6.2296	6.3081	6.3877	6.4684	6.6330	6.8019	6.9753
7	7.3230	7.4343	7.5474	7.6625	7.8983	8.1420	8.3938
8	8.4328	8.5830	8.7361	8.8923	9.2142	9.5491	9.8975
9	9.5593	9.7546	9.9545	10.1591	10.5828	11.0266	11.4913
10	10.7027	10.9497	11.2034	11.4639	12.0061	12.5779	13.1808
11	11.8633	12.1687	12.4835	12.8078	13.4864	14.2068	14.9716
12	13.0412	13.4121	13.7956	14.1920	15.0258	15.9171	16.8699
13	14.2368	14.6803	15.1404	15.6178	16.6268	17.7130	18.8821
14	15.4504	15.9739	16.5190	17.0863	18.2919	19.5986	21.0151
15	16.6821	17.2934	17.9319	18.5989	20.0236	21.5786	23.2760
16	17.9324	18.6393	19.3802	20.1569	21.8245	23.6575	25.6725
17	19.2014	20.0121	20.8647	21.7616	23.6975	25.8404	28.2129
18	20.4894	21.4123	22.3863	23.4144	25.6454	28.1324	30.9057
19	21.7967	22.8406	23.9460	25.1169	27.6712	30.5390	33.7600
20	23.1237	24.2974	25.5447	26.8704	29.7781	33.0660	36.7856
21	24.4705	25.7833	27.1833	28.6765	31.9692	35.7193	39.9927
22	25.8376	27.2990	28.8629	30.5368	34.2480	38.5052	43.3923
23	27.2251	28.8450	30.5844	32.4529	36.6179	41.4305	46.9958
24	28.6335	30.4219	32.3490	34.4265	39.0826	44.5020	50.8156
25	30.0630	32.0303	34.1578	36.4593	41.6459	47.7271	54.8645
26	31.5140	33.6709	36.0117	38.5530	44.3117	51.1135	59.1564
27	32.9867	35.3443	37.9120	40.7096	47.0842	54.6691	63.7058
28	34.4815	37.0512	39.8598	42.9309	49.9676	58.4026	68.5281
29	35.9987	38.7922	41.8563	45.2189	52.9663	62.3227	73.6398
30	37.5387	40.5681	43.9027	47.5754	56.0849	66.4388	79.0582
31	39.1018	42.3794	46.0003	50.0027	59.3283	70.7608	84.8017
32	40.6883	44.2270	48.1503	52.5028	62.7015	75.2988	90.8898
33	42.2986	46.1116	50.3540	55.0778	66.2095	80.0638	97.3432
34	43.9331	48.0338	52.6129	57.7302	69.8579	85.0670	104.1838
35	45.5921	49.9945	54.9282	60.4621	73.6522	90.3203	111.4348
36	47.2760	51.9944	57.3014	63.2759	77.5983	95.8363	119.1209
37	48.9851	54.0343	59.7339	66.1742	81.7022	101.6281	127.2681
38	50.7199	56.1149	62.2273	69.1594	85.9703	107.7095	135.9042
39	52.4807	58.2372	64.7830	72.2342	90.4091	114.0950	145.0585
40	54.2679	60.4020	67.4026	75.4013	95.0255	120.7998	154.7620
41	56.0819	62.6100	70.0876	78.6633	99.8265	127.8398	165.0477
42	57.9231	64.8622	72.8398	82.0232	104.8196	135.2318	175.9505
43	59.7920	67.1595	75.6608	85.4839	110.0124	142.9933	187.5076
44	61.6889	69.5027	78.5523	89.0484	115.4129	151.1430	199.7580
45	63.6142	71.8927	81.5161	92.7199	121.0294	159.7002	212.7435
46	65.5684	74.3306	84.5540	96.5015	126.8706	168.6852	226.5081
47	67.5519	76.8172	87.6679	100.3965	132.9454	178.1194	241.0986
48	69.5652	79.3535	90.8596	104.4084	139.2632	188.0254	256.5645
49	71.6087	81.9406	94.1311	108.5406	145.8337	198.4267	272.9584
50	73.6828	84.5794	97.4843	112.7969	152.6671	209.3480	290.3359

6. PRESENT VALUE OF AN ANNUITY: $a_{\overline{n}|i}$

n	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	4%	5%	6%
1	.9852	.9804	.9756	.9709	.9615	.9524	.9434
2	1.9559	1.9416	1.9274	1.9135	1.8861	1.8594	1.8334
3	2.9122	2.8839	2.8560	2.8286	2.7751	2.7232	2.6730
4	3.8544	3.8077	3.7620	3.7171	3.6299	3.5460	3.4651
5	4.7826	4.7135	4.6458	4.5797	4.4518	4.3295	4.2124
6	5.6972	5.6014	5.5081	5.4172	5.2421	5.0757	4.9173
7	6.5982	6.4720	6.3494	6.2303	6.0021	5.7864	5.5824
8	7.4859	7.3255	7.1701	7.0197	6.7327	6.4632	6.2098
9	8.3605	8.1622	7.9709	7.7861	7.4353	7.1078	6.8017
10	9.2222	8.9826	8.7521	8.5302	8.1109	7.7217	7.3601
11	10.0711	9.7868	9.5142	9.2526	8.7605	8.3064	7.8869
12	10.9075	10.5753	10.2578	9.9540	9.3851	8.8633	8.3838
13	11.7315	11.3484	10.9832	10.6350	9.9856	9.3936	8.8527
14	12.5434	12.1062	11.6909	11.2961	10.5631	9.8986	9.2950
15	13.3432	12.8493	12.3814	11.9379	11.1184	10.3797	9.7122
16	14.1313	13.5777	13.0550	12.5611	11.6523	10.8378	10.1059
17	14.9076	14.2919	13.7122	13.1661	12.1657	11.2741	10.4773
18	15.6726	14.9920	14.3534	13.7535	12.6593	11.6896	10.8276
19	16.4262	15.6785	14.9789	14.3238	13.1339	12.0853	11.1581
20	17.1686	16.3514	15.5892	14.8775	13.5903	12.4622	11.4699
21	17.9001	17.0112	16.1845	15.4150	14.0292	12.8212	11.7641
22	18.6208	17.6580	16.7654	15.9369	14.4511	13.1630	12.0416
23	19.3309	18.2922	17.3321	16.4436	14.8568	13.4886	12.3034
24	20.0304	18.9139	17.8850	16.9355	15.2470	13.7986	12.5504
25	20.7196	19.5235	18.4244	17.4131	15.6221	14.0939	12.7834
26	21.3986	20.1210	18.9506	17.8768	15.9828	14.3752	13.0032
27	22.0676	20.7069	19.4640	18.3270	16.3296	14.6430	13.2105
28	22.7267	21.2813	19.9649	18.7641	16.6631	14.8981	13.4062
29	23.3761	21.8444	20.4535	19.1885	16.9837	15.1411	13.5907
30	24.0158	22.3965	20.9303	19.6004	17.2920	15.3725	13.7648
31	24.6461	22.9377	21.3954	20.0004	17.5885	15.5928	13.9291
32	25.2671	23.4683	21.8492	20.3888	17.8736	15.8027	14.0840
33	25.8790	23.9886	22.2919	20.7658	18.1476	16.0025	14.2302
34	26.4817	24.4986	22.7238	21.1318	18.4112	16.1929	14.3681
35	27.0756	24.9986	23.1452	21.4872	18.6646	16.3742	14.4982
36	27.6607	25.4888	23.5563	21.8323	18.9083	16.5469	14.6210
37	28.2371	25.9695	23.9573	22.1672	19.1426	16.7113	14.7368
38	28.8051	26.4406	24.3486	22.4925	19.3679	16.8679	14.8460
39	29.3646	26.9026	24.7303	22.8082	19.5845	17.0170	14.9491
40	29.9158	27.3555	25.1028	23.1148	19.7928	17.1591	15.0463
41	30.4590	27.7995	25.4661	23.4124	19.9931	17.2944	15.1380
42	30.9941	28.2348	25.8206	23.7014	20.1856	17.4232	15.2245
43	31.5212	28.6616	26.1664	23.9819	20.3708	17.5459	15.3062
44	32.0406	29.0800	26.5038	24.2543	20.5488	17.6628	15.3832
45	32.5523	29.4902	26.8330	24.5187	20.7200	17.7741	15.4558
46	33.0565	29.8923	27.1542	24.7754	20.8847	17.8801	15.5244
47	33.5532	30.2866	27.4675	25.0247	21.0429	17.9810	15.5890
48	34.0426	30.6731	27.7732	25.2667	21.1951	18.0772	15.6500
49	34.5247	31.0521	28.0714	25.5017	21.3415	18.1687	15.7076
50	34.9997	31.4236	28.3623	25.7298	21.4822	18.2559	15.7619

Answers to odd-numbered exercises

GROUP 1, p. 18

1. $a^3 + a^2b - 2b^3 - 2ab^2$. **3.** $2x^2$. **5.** $2c^2 - cd + 2d + c$.
7. $2x^3 - 6x^2 + 5x - 2$. **9.** $3a + 6by - 5cy^2 - dy^3$. **11.** $2x^3 + 7x - 4$.
13. $-2x^3 + 2x^2 - x + 10$. **15.** $-4x^2 + 13x - 6$. **17.** 6. **19.** $x - 2y$.
21. $-m + 9n$. **23.** (a) -2; (b) -15. **25.** $-a + 5b - 6c$.
27. $6m - n + p$. **29.** $4x^2 - 3$. **31.** $3a^3 + a^2 - a$.

GROUP 2, p. 27

1. $-16a^3b^3$. **3.** $x^3y^2 - 2xy^3 + 4xy^2$. **5.** $3a^3 - a^2b - 20ab^2 + 14b^3$.
7. $a^3 + 8b^3$. **9.** $-m^6 + 2m^5 - 3m^4 + 4m^3 - 3m^2 + 2m - 1$.
11. $xyz + axy + axz + ayz + a^2x + a^2y + a^2z + a^3$. **13.** $a^5 - b^5$.
15. $a^8 + a^4 + 1$.
37. $a^4 - 2a^3b + a^2b^2$. **39.** $a^2x^2 - x^2y^2$. **41.** $a^2 + 2ac + c^2 - b^2$.
43. $12x^2 + 2x - 4$. **45.** $27m^3 - 54m^2n^2 + 36mn^4 - 8n^6$. **47.** $x^8 - 1$.
49. $a^2 + b^2 + c^2 + d^2 - 2ab + 2ac - 2ad - 2bc + 2bd - 2cd$.

GROUP 3, p. 36

1. $-2x^2yz$. **3.** $2ax - 4by$. **5.** $2x - 3y$. **7.** $a - 3b$.
9. $m^3 - m^2n + mn^2 - n^3$. **11.** $x^4 - x^3y + x^2y^2 - xy^3 + y^4$.
13. $x^2 - 2xy - 2y^2$. **15.** $2a^2 - 3ab + b^2$. **17.** $2x - y + 2$.
19. $Q = 2a^3 + 2a^2 - a + 1$, $R = -6$. **21.** $a^2 - 2ab + b^2$.
27. $x + 2y$. **31.** $2x + y - 2$. **33.** -51. **35.** $x^3 + x - 3$.

GROUP 4, p. 42

1. $2xy^2(x^2 - 3y)$. **3.** $2b^2(2m + 3n)^2$. **5.** $(x + y + a)(x + y - a)$.
7. $(m - n + b)(m - n - b)$. **9.** $(3a - 2)(2a + 3)$. **11.** $(4x - 3)(3x - 5)$.
13. $(2m - 3n)(5m + n)$. **15.** $(x + y - 2)(x + y + 3)$. **17.** $(x + 3)(x - 2y)$.
19. $2a^2(2x - y)(m + 2n)$. **21.** $(2x - y)^3$.
23. $(ab^2 + 3c^2d)(a^2b^4 - 3ab^2c^2d + 9c^4d^2)$. **25.** $(1 + my)(1 + y)(1 - y)$.
27. $(x^2 + x + 1)(x^2 - x + 1)(x^4 - x^2 + 1)$.
29. $(x + y + z)(x + y - z)(x + z - y)(y + z - x)$.

31. $(2x - 1)(x + 2)(3x - 2)$. **33.** $(a + 2b)(a - b)(3a - 2b)$.
35. $(x + 1)(x - 2)(x + 3)$. **37.** $(x^4 - 16)(x + 3)$.
39. $m(m - 1)(2m + 3)(m + n)$.

GROUP 5, p. 48

1. $\frac{a}{a-b}$. **3.** $\frac{x-y}{x^2 - xy + y^2}$. **5.** $\frac{x}{1-x}$. **7.** $x + 4 - \frac{3x+3}{x^2+1}$.
9. $\frac{x^3+1}{x-1}$. **11.** $\frac{3x^2-1}{x(x^2-1)}$. **13.** $\frac{3x}{x^2-4}$. **15.** $\frac{1}{a^4 + a^2 + 1}$.
17. 0. **19.** 0. **21.** $\frac{3ax}{2by}$. **23.** $x - a$. **25.** $\frac{x-1}{x+2}$.
27. 1. **31.** $\frac{6}{7}$. **33.** 1. **35.** $a - x$. **37.** $\frac{2xy}{x^2+y^2}$.
39. $\frac{x-1}{x+2}$. **41.** $1 - x$. **43.** x. **45.** $\frac{a}{2x^2}$.

GROUP 6, p. 54

3. 8. **5.** $\frac{1}{5}$. **7.** $\frac{27}{64}$. **9.** $\frac{4}{3}$. **11.** $\frac{2}{a^2}$. **13.** m. **15.** ma.
17. x^2. **19.** $x - y$. **21.** $x + 2 + x^{-1}$. **23.** $x - 3x^{2/3}y^{1/3} + 3x^{1/3}y^{2/3} - y$.
25. $a - b$. **27.** $m^3 - 1$. **29.** $x^{2/3} - x^{1/3}y^{1/3} + y^{2/3}$.
31. $a^{1/3} + a^{1/6}x^{1/3} + x^{2/3}$. **33.** x^4y^{14}. **35.** $\frac{x-y}{x^2(x+2y)}$.
37. $2(4 - x^2)^{1/2}$. **39.** $\frac{2xy}{(x+y)^2}$.

GROUP 7, p. 60

5. $-3b\sqrt[3]{b^2}$. **7.** $3a^2x\sqrt{5ax}$. **9.** $\frac{5a}{2}\sqrt{6a}$. **11.** $\sqrt{2a}$.
13. $2\sqrt{3} - 3\sqrt{2}$. **15.** $\sqrt[3]{6} + \sqrt[3]{2}$. **17.** $12\sqrt{3}$. **19.** $\sqrt{5}$.
21. $\sqrt[12]{32}$. **23.** $\sqrt{6} + 2 - \frac{1}{3}\sqrt{15}$. **25.** $\sqrt[3]{2} - \sqrt[6]{432} - \sqrt[6]{2}$.
27. $30 - 12\sqrt{6}$. **29.** $9\sqrt{3} - 11\sqrt{2}$. **31.** $2\sqrt{6}$. **33.** 0.
35. $6 + 2\sqrt{6}$. **37.** $5 + 2\sqrt{6}$. **39.** $\frac{2\sqrt{3} + 3\sqrt{2} - \sqrt{30}}{12}$.
41. $\frac{1 - \sqrt{1 - a^4}}{a^2}$. **43.** $\frac{\sqrt{10} - \sqrt{35}}{5}$. **45.** $\frac{\sqrt{a}(x+1)}{b-a}$.

GROUP 8, p. 63

11. $(x - 2y + 3)(3x + y - 2)$. **15.** $a(x - y)$; $a(x - y)(x + y)(x + 2y)$.
21. (a) $\sqrt{5}$; (b) $\sqrt{6}$. **23.** 3.7320508. **27.** $\frac{1 + \sqrt[3]{a} + \sqrt[3]{a^2}}{1 - a}$.
29. $\sqrt[3]{9} + \sqrt[3]{6} + \sqrt[3]{4}$. **31.** $3 + 2\sqrt{5}$. **33.** $-\sqrt{ab}$.

GROUP 9, p. 70

1. (a) $h = \frac{3V}{\pi r^2}$; (b) $r = \sqrt{\frac{3V}{\pi h}}$. **3.** $d = \sqrt{2A}$. **5.** 1; 7; $\frac{7}{9}$.
9. 4; 5; 3. **11.** $3 + 2\sqrt{2}$. **13.** 0; 0; 8; 8. **15.** $5 + 2\sqrt{6}$.
17. $\frac{-1}{(x + h + 1)(x + 1)}$. **25.** $\frac{6x + 1}{2x + 3}$.

GROUP 10, p. 78

29. −1, 2. **31.** 0.7, −2.7. **33.** 0, 1, 2.

GROUP 11, p. 86

1. 1. **3.** 5. **5.** 7. **7.** a. **9.** m. **11.** b. **13.** 0. **15.** 5. **17.** 2.
19. $\frac{2rs}{r + s}$. **21.** $y = \frac{6 - 3x}{2}$, $x = \frac{6 - 2y}{3}$. **23.** $y = \frac{ab - bx}{a}$, $x = \frac{ab - ay}{b}$.
25. $\frac{A - P}{Pr}$. **27.** $\frac{s - s_0 - \frac{1}{2}gt^2}{t}$. **31.** $\frac{9}{2}$.

GROUP 12, p. 88

1. 9, 12. **3.** 6, 7, 8. **5.** 9, 15. **7.** 50. **9.** 16, 24, 32.
11. 20 ft. **13.** $1\frac{1}{3}$ hrs. **15.** 24 hrs. **17.** 3 hrs.
19. 42 mi. **21.** 15 min. **23.** 2 of 60%, 4 of 90%.
25. 37.5 lb. **27.** $49\frac{1}{11}$ min. past 3. **29.** A, 6 days; B, 3 days; C, 2 days.

GROUP 13, p. 97

1. (1, 1). **3.** (3, −1). **5.** (0, 0). **7.** No solution.
9. No solution. **11.** $\left(-2, \frac{4}{3}\right)$. **13.** $(2a, b)$. **15.** (4, 3, 2).
17. (2, 3, 6). **21.** $\frac{3}{8}$. **23.** 72. **25.** 300 mi., 50 mi. per hr.
27. $a = 2$, $b = 3$. **29.** A, 3 days; B, 4 days; C, 5 days.

GROUP 14, p. 104

1. 1, 2. **3.** $\frac{1}{3}$, −1. **5.** 2, 3. **7.** 4, −3. **9.** 5, −2.
11. $1 \pm \sqrt{2}$. **13.** $\frac{2 \pm \sqrt{5}}{3}$. **15.** (0, −8). **17.** $2 \pm 3i$. **19.** $3 \pm i$.
21. $\frac{a}{b}, \frac{b}{a}$. **23.** $b \pm \sqrt{a}$. **27.** 15 ft by 10 ft. **29.** 3 hrs., 6 hrs.
31. 100. **33.** 2 secs. **35.** 5 in., 7 in.

GROUP 15, p. 109

1. Real and unequal; -1; -6. **3.** Conjugate complex; 2; 3. **5.** Real and unequal; 4; 1. **7.** 4. **9.** $-8, 4$. **11.** 5. **13.** $x^2 - 7x + 12 = 0$. **15.** $x^2 - 2 = 0$. **17.** $x^2 - 2x - 4 = 0$. **19.** $(x - 2)(x - 5)$. **21.** $(x + 1 + 2i)(x + 1 - 2i)$. **23.** $k = 1$; -2. **25.** 3. **27.** 1. **29.** $2, -\frac{10}{9}$. **31.** $a = 2, b = -1$. **33.** -2.

GROUP 16, p. 114

1. $\pm 1, \pm 4$. **3.** 4. **5.** $4, \frac{1}{4}$. **7.** $1, 1, -3 \pm 2\sqrt{2}$. **9.** $2, -1, \frac{-1 \pm \sqrt{33}}{4}$. **11.** 5. **13.** $3, -5$. **15.** 1. **17.** No solution. **19.** 2. **21.** $\frac{16}{25}$. **23.** $1, -3$. **25.** $16x^2 + 25y^2 = 400$.

GROUP 17, p. 120

1. min. = 3 for $x = -2$. **3.** min. = 0 for $x = 3$. **5.** max. = 4 for $x = 1$. **7.** Pos. when $x < 1$ and $x > 4$; neg. when $1 < x < 4$; zero when $x = 1, 4$; min. $= -\frac{9}{4}$ when $x = \frac{5}{2}$. **9.** Pos. for all values of x except 1; zero when $x = 1$; min. = 0 for $x = 1$. **11.** Pos. for all real values of x; min. $= \frac{3}{4}$ when $x = \frac{1}{2}$. **15.** $\frac{1}{2}$. **17.** Each leg 7 in. long. **19.** 25 ft by 50 ft. **25.** $ax^2 - 6ax + 9a + 5, a > 0$.

GROUP 18, p. 125

1. $(4, 2), \left(\frac{9}{4}, -\frac{3}{2}\right)$. **3.** $(1, 2), (4, -4)$. **5.** $(1, -1), (1, -1)$. **7.** $\left(\frac{5 + \sqrt{7}i}{2}, \frac{5 - \sqrt{7}i}{2}\right), \left(\frac{5 - \sqrt{7}i}{2}, \frac{5 + \sqrt{7}i}{2}\right)$. **9.** $(1, 1), (-1, -1)$. **11.** $-2, -10$. **13.** $\left(\frac{2\sqrt{15}}{5}, \frac{2}{5}\sqrt{10}\right), \left(\frac{2\sqrt{15}}{5}, -\frac{2}{5}\sqrt{10}\right), \left(\frac{-2\sqrt{15}}{5}, \frac{2}{5}\sqrt{10}\right), \left(\frac{-2\sqrt{15}}{5}, -\frac{2}{5}\sqrt{10}\right)$. **15.** $\left(\frac{6\sqrt{13}}{13}, \frac{6\sqrt{13}}{13}\right), \left(\frac{6\sqrt{13}}{13}, \frac{-6\sqrt{13}}{13}\right), \left(\frac{-6\sqrt{13}}{13}, \frac{6\sqrt{13}}{13}\right), \left(\frac{-6\sqrt{13}}{13}, \frac{-6\sqrt{13}}{13}\right)$. **17.** $(4, 0), (4, 0), (-4, 0), (-4, 0)$. **19.** $\left(\frac{\sqrt{10}}{2}, \frac{\sqrt{6}}{2}i\right), \left(\frac{\sqrt{10}}{2}, \frac{-\sqrt{6}}{2}i\right), \left(\frac{-\sqrt{10}}{2}, \frac{\sqrt{6}}{2}i\right) \left(\frac{-\sqrt{10}}{2}, \frac{-\sqrt{6}}{2}i\right)$. **21.** 5, 2. **23.** 15 ft by 25 ft. **25.** $\frac{p}{m}$.

GROUP 19, p. 130

1. (2, 1), (−2, −1), (1, 2), (−1, −2). **3.** $(2, -2), (-2, 2), (0, 2\sqrt{2}), (0, -2\sqrt{2})$.

5. $(3, 5), (-3, -5), \left(\frac{5}{3}, \frac{13}{3}\right), \left(-\frac{5}{3}, -\frac{13}{3}\right)$. **15.** (2, 1), (1, 2), (−1, −2), (−2, −1).

17. $(2, -1), (-1, 2), \left(\frac{1}{2} + \frac{\sqrt{15}}{2} i, \frac{1}{2} - \frac{\sqrt{15}}{2} i\right), \left(\frac{1}{2} - \frac{\sqrt{15}}{2} i, \frac{1}{2} + \frac{\sqrt{15}}{2} i\right)$.

19. (4, −3), (−4, 3), (−3, 4), (3, −4). **21.** (4, 2), (−2, −4).

23. (1, 5), (−1, −5), (14, −8), (−14, 8). **25.** $(0, 0), \left(\frac{1}{2}, \frac{1}{6}\right), \left(\frac{1}{6}, \frac{1}{2}\right)$.

27. $(1, 5), \left(\frac{39}{29}, \frac{25}{29}\right)$. **29.** 2, 2.

GROUP 21, p. 138

17. $a > -1, a \neq 1$.

GROUP 22, p. 145

1. $x > 4$. **3.** $x > 2$. **5.** $x < -2$.
7. Pos. when $-1 < x < 2$; neg. when $x < -1, x > 2$; zero when $x = -1, 2$.

9. Pos. for all values of x. **11.** $x > 3, x < -2$. **13.** $x > \frac{1}{4}, x < -3$.

15. All values of x. **17.** All values of x. **19.** All values of x except 4.
21. $x \geq 7$. **23.** $x \geq 4, x \leq -4$. **25.** $k > 4, k < -4$. **27.** $-4 < k < 0$.
29. $-5 < k < 5$. **31.** $-2 < x < 0, x > 1$. **33.** $1 < x < 2, x < -1$.
35. $x < -2, 1 < x < 4$. **37.** $x > 1$. **39.** $-3 < x < 1$. **41.** $-2 < x < 1$.

43. $x > 4$. **45.** $-\frac{5}{4} < x < -\frac{1}{2}, x > 1$. **47.** $x > -\frac{1}{2}, -1 > x > -2$.

49. $0 < x < 1, x > 2$.

GROUP 23, p. 148

1. $-2 < x < 2$. **3.** $1 < x < 3$. **5.** $4 < x < 6$. **7.** $0 > x > -4$.

13. $x > 3$. **15.** $1 \leq x < 2$. **17.** $x > -\frac{3}{4}$. **19.** $x > 4$. **21.** $x > 6, 2 \leq x < 3$.

GROUP 25, p. 161

1. $81a^4 - 108a^3b + 54a^2b^2 - 12ab^3 + b^4$.
3. $x^6 + 18x^5y + 135x^4y^2 + 540x^3y^3 + 1215x^2y^4 + 1458xy^5 + 729y^6$.

5. $x^8 + 4x^{13/2} + 6x^5 + 4x^{7/2} + x^2$. **7.** $\frac{a^4}{16} - a^2 + 6 - \frac{16}{a^2} + \frac{16}{a^4}$.

9. $\frac{x^2}{y^4} - \frac{4x}{y^2} + 6 - \frac{4y^2}{x} + \frac{y^4}{x^2}$. **11.** $a^6 - 4a^3 + 6 - 4a^{-3} + a^{-6}$.

13. $a^3 + 3a^2b + 3ab^2 + b^3 - 3a^2c - 6abc - 3b^2c + 3ac^2 + 3bc^2 - c^3$.

15. $128a^7 - 448a^6b + 672a^5b^2 - 560a^4b^3$. **17.** $a^9 - 3a^8b + 4a^7b^2 - \frac{28}{9}a^6b^3$.

19. $x^6 - 12x^{11/2}y^{1/2} + 66x^5y - 220x^{9/2}y^{3/2}$. **21.** $1 - x + x^2 - x^3$.

23. $1 + 2x + 3x^2 + 4x^3$. **25.** $1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6$. **29.** 1.04060401.

31. 0.995. **39.** $792x^{5/2}y^{7/2}$. **41.** $\frac{231}{16}a^5x^6$. **43.** -252.

45. $\frac{231}{16}a^6b^6, \frac{231}{32}a^5b^5$. **47.** 8th term $= \frac{1215y^4}{2x^4}$. **49.** 5th term $= 1820y^{-20/3}$.

GROUP 26, p. 168

1. $x = 2, y = -3$. **3.** $x = 3, y = -1$. **5.** $(2, -1), (-2, 1)$.

7. $\left(0, -\frac{1}{2}\right), \left(2, \frac{1}{2}\right)$. **9.** $4 - i$. **11.** $-1 + 5i$. **13.** $3i$. **15.** ai. **17.** 13.

19. $6 + 8i$. **21.** $-4 - 2\sqrt{6}$. **23.** -4. **25.** $-i$. **27.** $\frac{1}{5} + \frac{2}{5}i$.

29. $1 - 2i$. **31.** $-2 + i$. **33.** $\frac{1}{2} + \frac{1}{2}i$.

GROUP 27, p. 175

11. $1 + i$. **13.** $5 - i$. **15.** $-1 + i$. **17.** $8 + 2i$. **19.** $2 - 3i$.
21. $1 - 4i$. **23.** $-4i$. **25.** $r = 4, \theta = 120°$. **27.** $r = 2, \theta = 210°$.
29. $r = 7, \theta = 180°$. **31.** $r = 8, \theta = 240°$. **33.** $6i$. **35.** $-\sqrt{3} - i$.

37. $\frac{3}{4} + \frac{3}{4}\sqrt{3}i$. **39.** -2.

GROUP 28, p. 181

1. $4\sqrt{2} + 4\sqrt{2}i$. **3.** $27i$. **5.** $25(\cos 80° + i \sin 80°)$. **7.** $-8i$.

9. $-\frac{1}{4} + \frac{1}{4}i$. **11.** $-i$. **13.** -8. **15.** $-128 + 128\sqrt{3}i$.

17. $-16 - 16\sqrt{3}i$. **19.** $-3, \frac{3}{2} \pm \frac{3\sqrt{3}}{2}i$. **21.** $r = \sqrt{2}, \theta = 45°, 165°, 285°$.

23. $1 \pm i, -1 \pm i$. **25.** $r = 2, \theta = 0°, 72°, 144°, 216°, 288°$.
27. $r = \sqrt{3}, \theta = 15°, 75°, 135°, 195°, 255°, 315°$.
29. $r = 2, \theta = 15°, 60°, 105°, 150°, 195°, 240°, 285°, 330°$.
31. $r = 1, \theta = 30°, 70°, 110°, 150°, 190°, 230°, 270°, 310°, 350°$.

33. $\pm 1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i, \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. **35.** $\pm 1, \pm i$. **37.** $3, -\frac{3}{2} \pm \frac{3\sqrt{3}}{2}i$.

GROUP 29, p. 188

27. $r = 4.196$, $\theta = 17° 35'$. **29.** $u = \dfrac{x}{x^2 + y^2}$, $v = \dfrac{-y}{x^2 + y^2}$.

GROUP 30, p. 195

1. 14. **3.** −5. **5.** −2. **7.** $y = 3x - \dfrac{6}{x}$. **9.** $y = 2x^3 - 3x^2 + 5x$.
19. 5 secs. **21.** 2 secs. **23.** 400 lb. **25.** 16.7% decr. **27.** 17.4% decr.
29. 5.8% incr.

GROUP 32, p. 206

1. 42; 242. **3.** −17, −56. **5.** $\dfrac{29}{2}$; 52. **7.** $a_n = -16$, $s_n = -44$.
9. $n = 14$; $a_n = -15$. **11.** $n = 9$, $d = -5$. **13.** $n = 17$, $a_n = -3$. **15.** 63.
17. −2. **19.** −2, 0, 2, 4, 6. **21.** $-\dfrac{28}{3}, -\dfrac{20}{3}, -4, -\dfrac{4}{3}, \dfrac{4}{3}$. **23.** $d = 1$, $a_6 = 0$.
25. $a_7 = -4$, $s_{12} = -30$. **29.** n^2. **31.** $2n^2 + n$. **33.** 5. **35.** 1, 0.
37. $a_1 + \dfrac{n-2}{2}d$, $a_1 + \dfrac{n}{2}d$. **39.** $-2n$. **43.** 576 ft; 176 ft. **45.** 1357.

GROUP 33, p. 211

1. 1024; 2046. **3.** 4096; 5461. **5.** $\dfrac{3}{2}$; $94\frac{1}{2}$. **7.** $s_6 = \dfrac{135}{243}$; $n = 6$.
9. $r = 2$; $s_6 = 126$. **11.** $a_1 = 5$; $a_7 = 320$. **13.** 4, 1, $\dfrac{1}{4}$. **15.** $\dfrac{1}{4}, \dfrac{1}{2}$, 1, 2, 4.
17. xy. **19.** $r = \dfrac{1}{2}$, $a_1 = 12$. **21.** $a_7 = 729$, $s_6 = 364$. **23.** $s_n = na_1$.
29. $\left(\dfrac{9}{10}\right)^8$. **31.** $12\frac{73}{1296}$ gals. **33.** 2, 8.

GROUP 34, p. 214

1. $\dfrac{4}{9}$. **3.** $\dfrac{3}{4}, \dfrac{3}{2}$. **5.** $\dfrac{8}{5}, \dfrac{4}{3}, \dfrac{8}{7}$. **7.** $\dfrac{7}{2}, \dfrac{7}{3}, \dfrac{7}{4}, \dfrac{7}{5}, \dfrac{7}{6}$. **9.** $\dfrac{x^2 - y^2}{x}$. **11.** 3, 5.
13. 12. **15.** 2, $\dfrac{8}{3}$, 4; $-\dfrac{23}{2}, -\dfrac{46}{3}$, −23.

GROUP 35, p. 221

1. 24. **3.** $\dfrac{3}{2}(3 + \sqrt{3})$. **5.** $\dfrac{5}{4}(\sqrt{5} + 1)$. **7.** $\dfrac{7}{9}$. **9.** $\dfrac{35}{99}$. **11.** $\dfrac{41}{333}$.
13. $\dfrac{1489}{3300}$. **19.** $37\frac{1}{2}$ ft. **21.** $\dfrac{1}{16}$. **27.** 11.

GROUP 36, p. 229

5. 7, −17. **7.** 40, −12. **9.** $-\frac{5}{9}$, −0.8291. **11.** $x^2 + 2x + 3$, −8.
13. $x^5 + x^4 + x + 1$, −1. **15.** $4x^3 - 2x^2 - 2x + 4$, 5. **17.** No. **19.** Yes.
21. Yes. **23.** No. **25.** Yes. **27.** $x - 2$, $x + 3$. **29.** 3, −2.
31. $x^3 - 3x^2 + 3x - 2$; 5. **33.** −4. **35.** −5. **37.** $a = 2$, $b = -1$.

GROUP 37, p. 234

1. 1, 2, 3. **3.** 0, 4, −2. **5.** ±1, ±2. **7.** 4, −3, $2 < x < 3$, $-1 < x < 0$.
9. 2, −1, $5 < x < 6$, $-4 < x < -3$. **11.** ±2. **13.** 1, 1, 1, −2, −2.
19. $1 < x < 2$, $x > 3$. **21.** $x > 1$, $x < -2$. **23.** $-2 < x < 1$, $x > 3$.

GROUP 38, p. 238

1. $x^3 - 2x^2 - x + 2 = 0$. **3.** $x^4 - x^3 - 16x^2 + 4x + 48 = 0$.
5. $x^3 - 3x^2 + 2 = 0$. **7.** $x^4 - 6x^3 + 8x^2 + 2x - 1 = 0$.
9. $x^5 - 2x^4 - 6x^3 + 20x^2 - 19x + 6 = 0$. **11.** $x^4 + x^3 - 11x^2 + 35x - 50 = 0$.
13. −1, −2. **15.** $2 \pm i$. **17.** $\pm\sqrt{3}$. **19.** $\pm 2i$. **21.** $2 \pm \sqrt{3}$.
23. $A = 2$, $B = 3$. **25.** $A = -1$, $B = 1$, $C = -1$.

GROUP 39, p. 240

1. $1 + i$, −3. **3.** $2 - i$, 1, 1. **5.** $-\sqrt{5}$, −1. **7.** $3 - \sqrt{2}$, 1, 2.
9. $-i$, $1 + 2i$, 5. **11.** 0, $2 - \sqrt{2}$, $2 - 2i$. **13.** $x^3 - 4x^2 - 2x + 20 = 0$.
15. $x^4 - 4x^3 + 24x^2 - 16x + 80 = 0$. **17.** $x^4 - 3x^3 - 9x^2 + 25x - 6 = 0$.
19. $(x - 2)(x^2 + 5x + 7)$. **21.** $(x + 1)(x - 4)(x^2 + x + 1)$.
23. $(2x - 1)(x - 2)(x^2 - 2x - 1)$.

GROUP 40, p. 244

1. 1 pos., 1 neg., 2 complex. **3.** 2 pos., 1 neg.; 1 neg., 2 complex. **5.** 6 complex.
7. 1 zero, 4 complex. **9.** ±1, 6 complex. **11.** 8 complex.
13. 2 zero, 3 pos.; 2 zero, 1 pos., 2 complex.
15. 3 pos., 2 neg., 4 complex; 1 pos., 2 neg., 6 complex; 3 pos., 6 complex; 1 pos., 8 complex.

GROUP 41, p. 249

3. 4, −3, $\frac{1}{3}$. **5.** $\frac{1}{2}, \frac{2}{3}$, −6. **7.** 3, −5, $-\frac{1}{3}, \frac{2}{3}$. **9.** 0, $\pm\frac{1}{2}, \frac{1 \pm \sqrt{5}}{2}$.
11. 0, 0, −1, 2, $\pm\sqrt{2}i$. **13.** 1, $\frac{1}{3}$, $\pm 3i$. **15.** −3, $\frac{1}{6}, \frac{1 \pm \sqrt{7}i}{2}$.
17. 4, $-\frac{1}{2}, \pm\frac{5}{2}i$. **19.** −2. **21.** 3. **23.** −1, $\frac{2}{3}$. **29.** 3 ft.

GROUP 42, p. 255

1. 2.6. **3.** 1.1. **5.** 2.5. **7.** 3.4. **9.** −1.24.
13. $x^3 + x^2 - 26x + 24 = 0.$ **15.** $x^3 - 6x^2 - 12x + 112 = 0.$
19. $2x^4 - 6x^3 - 7x^2 + 12 = 0.$ **21.** $x^4 + 3x^3 + 2x^2 + x + 1 = 0.$
23. $x^3 - 10x^2 + 9x + 56 = 0.$ **25.** $x^3 - x^2 - 9x + 9 = 0.$
29. $3x^3 + 5x^2 - 34x - 24 = 0.$ **31.** $x^4 + 6x^3 + 11x^2 + 10x + 1 = 0.$
33. $2x^3 + 3.06x^2 + 1.0606x - 0.989698 = 0.$ **35.** $3x^3 - 13x^2 - 18x + 40 = 0.$

GROUP 43, p. 259

1. 3.21. **3.** 1.25. **5.** 3.19. **7.** 1.095. **9.** 3.264. **11.** 2.157. **17.** 0.28.
19. 4.464. **21.** 2.736. **23.** 1.913. **25.** −3.271. **27.** 1.933. **29.** 1.075, 2.9.

GROUP 44, p. 262

1. $-\frac{1}{2}, 1, \frac{5}{2}.$ **3.** 2, 3, 4; $k = 26.$ **5.** $2, -2, \frac{1}{4}.$ **7.** −3, −3, 4.

9. $3, \frac{1}{3}, -9.$ **11.** $-\frac{1}{2}, -1, -3.$ **13.** 1, 3, −2. **15.** $\frac{1}{2}, \frac{1}{2}, -4, -4.$

17. $\frac{1}{3}, -\frac{1}{3}, 6, 1.$ **21.** $\frac{3}{2}.$ **23.** $ab = c.$

GROUP 45, p. 270

1. $\frac{2}{x-2} + \frac{1}{x+4}.$ **3.** $\frac{2}{x+3} - \frac{1}{x-3}.$ **5.** $\frac{2}{x+2} - \frac{1}{x-3} + \frac{2}{x+5}.$

7. $\frac{3}{x-3} + \frac{2}{2x+1} - \frac{5}{x-1}.$ **9.** $x + 1 + \frac{3}{x-2} + \frac{2}{x+3}.$ **11.** $\frac{3}{x+1} - \frac{4}{(x+1)^2}.$

13. $\frac{3}{x} + \frac{1}{x^2} - \frac{2}{x^3} + \frac{6}{x+5}.$ **15.** $\frac{4}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1} + \frac{3}{(x+1)^2}.$

17. $\frac{1}{x+2} + \frac{2}{(x+2)^2} + \frac{1}{x-1} - \frac{2}{(x-1)^2}.$ **19.** $2 + \frac{3}{x} + \frac{2}{x^2} + \frac{1}{x-1} - \frac{1}{(x-1)^2}.$

GROUP 46, p. 273

1. $\frac{2}{x-1} + \frac{x-3}{x^2+1}.$ **3.** $\frac{2x}{x^2+1} - \frac{4}{x^2+2}.$ **5.** $\frac{x-1}{x^2+2} - \frac{x-3}{x^2+3}.$

7. $\frac{4}{x+1} + \frac{2}{x-1} - \frac{2x+1}{x^2+x+3}.$ **9.** $2 + \frac{1}{x} - \frac{2}{x^2} + \frac{x-1}{x^2+x+3}.$

11. $\frac{x+1}{x^2+x+1} + \frac{x-1}{(x^2+x+1)^2}.$ **13.** $\frac{3}{x} - \frac{4}{x^2} + \frac{2x-1}{x^2-x+1} - \frac{x+3}{(x^2-x+1)^2}.$

15. $\frac{2}{x+1} - \frac{1}{(x+1)^2} + \frac{3}{x^2-x+1} + \frac{2x}{(x^2-x+1)^2}.$

17. $\frac{3}{x-1} + \frac{1}{(x-1)^3} + \frac{2x-1}{x^2+x+1} + \frac{x+2}{(x^2+x+1)^2}.$

19. $2x + 1 + \frac{3}{x^2+3} + \frac{x+1}{x^2+1} - \frac{2}{(x^2+1)^2} + \frac{2x}{(x^2+1)^3}.$

GROUP 47, p. 277

3. 16. **5.** 24. **7.** 1980. **9.** 20; 25. **11.** 60; 120; 120. **13.** 4; 8; 2^n. **15.** f^n. **17.** 504. **19.** 421,200. **21.** 13,353,984. **23.** 6720. **25.** 1190.

GROUP 48, p. 281

5. (a) 5040; (b) 7. **7.** 10. **9.** 2. **11.** 720. **13.** 51,840. **15.** 560. **17.** 462. **19.** $\frac{(p+q)!}{p!\,q!}$. **21.** 325. **23.** 240. **25.** 720. **27.** 5760. **29.** 2520. **31.** 720. **33.** (a) 5040; (b) 1440. **35.** (a) 720; (b) 240.

GROUP 49, p. 285

3. (a) 70; (b) 21. **5.** 8. **7.** 9. **9.** 2. **13.** 1365. **15.** 1001. **17.** 11. **19.** 720. **21.** (a) 495; (b) 330; (c) 210. **23.** 3150. **25.** 861. **27.** 36. **29.** 714.

GROUP 50, p. 293

7. 5775. **9.** $u_1^3 + u_2^3 + \cdots + u_n^3$. **11.** $1 + 3 + 5 + 7$. **15.** 15. **17.** (a) 16; (b) 16. **19.** (a) 20; (b) 42; (c) 63. **21.** 70.

GROUP 51, p. 301

3. 3 to 2. **5.** $\frac{3}{10}$. **7.** $\frac{1}{6}$; 5 to 1. **9.** $\frac{1}{4}$. **11.** (a) $\frac{1}{4}$; (b) $\frac{1}{2}$; (c) $\frac{3}{4}$. **13.** $\frac{25}{66}$. **15.** $\frac{60}{143}$. **17.** 326. **19.** \$6.50. **21.** $\frac{1}{12}$. **23.** $\frac{20}{77}$. **25.** $\frac{5}{108}$. **27.** $\frac{5}{12}$. **29.** \$5.50. **31.** 66 cents. **33.** (a) $\frac{11}{850}$; (b) $\frac{22}{425}$. **35.** $\frac{1}{126}$. **37.** $\frac{2197}{20825}$. **39.** $\frac{94}{4165}$.

GROUP 52, p. 307

9. $\frac{25}{216}$. **11.** $\frac{1}{72}$. **13.** $\frac{14}{15}$. **19.** (a) $\frac{2}{9}$; (b) $\frac{4}{15}$. **21.** $\frac{8}{15}$. **23.** $\frac{9}{16}$. **52.** $\frac{9}{16}$. **27.** $\frac{5}{12}$. **29.** $\frac{4}{17}$. **31.** (a) 0.72; (b) 0.02; (c) 0.18; (d) 0.08. **33.** (a) $\frac{1}{4}$; (b) $\frac{11}{24}$; (c) $\frac{1}{4}$; (d) $\frac{1}{24}$. **35.** $\frac{3}{4}$. **37.** A, $\frac{6}{11}$; B, $\frac{5}{11}$. **39.** \$20, \$10, \$5.

GROUP 53, p. 318

3. $\frac{5}{7776}$. **7.** (a) $\frac{216}{625}$; (b) $\frac{16}{625}$. **9.** 0.2646. **11.** $\frac{7}{128}$. **13.** $\frac{13}{3888}$.
15. $\frac{459}{512}$. **17.** $\frac{8585216}{9765625}$. **19.** $\frac{1053}{3125}$. **23.** $\frac{1}{2}$. **25.** $\frac{5}{324}$. **27.** 5; $\frac{63}{256}$.
29. 4; $\frac{14080}{59049}$. **31.** $\frac{63}{256}$.

GROUP 54, p. 326

1. -14. **3.** -28. **5.** $-3ax$. **7.** $x^2 - 6x - 3$. **9.** 2, -3. **11.** $(5, -2)$
13. $(-3, 4)$. **15.** No solution.

GROUP 55, p. 334

1. 73. **3.** 107. **5.** 16. **7.** 48. **9.** 2, 3. **11.** (3, 1, 2). **13.** $(2, 6, -2)$
15. (0, 0, 0). **33.** $x - 2y - 2 = 0$. **35.** 13.

GROUP 56, p. 344

9. 7. **11.** $1 + x^2 + y^2 + z^2$. **13.** -24. **15.** 3. **17.** 1288.
23. $6x^2 + 6y^2 - 32x - 25y - 34 = 0$. **29.** 40. **31.** $(a - b)(b - c)(c - a)$.
33. $(a - b)(a - c)(a - d)(b - c)(b - d)(c - d)$.

GROUP 57, p. 356

3. $(3, -1, 2)$. **5.** $(-3, 0, 1)$. **7.** $(1, -1, 3, 2)$. **9.** $(2, -1, 0, 2, 0)$.
15. $1:2:-1:3$. **17.** $(1, 2, -1, 1)$.
19. 1, 12, 5; 2, 10, 6; 3, 8, 7; 4, 6, 8; 5, 4, 9; 6, 2, 10. **21.** (3, 2, 1).
23. $k = 1$, $(2, 2, -1)$.

GROUP 58, p. 362

1. $\log_2 16 = 4$. **3.** $\log_{1/8} \frac{1}{4} = \frac{2}{3}$. **5.** $\log_x z = y$. **7.** $10^2 = 100$.

9. $10^{-1} = 0.1$. **11.** $8^{2/3} = 4$. **13.** 3. **15.** 4. **17.** 10. **19.** $\frac{3}{2}$.

21. 64. **25.** $x = 1 + \log_{10} y$.

GROUP 59, p. 366

9. $\log_b (x + 1) + \log_b (x - 1) - \log_b (x + 2) - \log_b (x - 2)$.

11. $\log_b x + 2 \log_b (x + 2) - 4 \log_b (x - 2)$. **13.** $\frac{1}{2} [\log_b (x^2 + 1) - \log_b (x^2 + 2)]$.

15. 4. **17.** 0.72. **19.** (a) 3; (b) 4. **21.** $x = \log_b y - 2$.

23. $x = \log_b \frac{y - 1}{y}$. **25.** $x = \log_b \frac{y - 1}{y + 1}$. **27.** $x = \frac{b^y}{b^y - 1}$. **29.** $x = \frac{2}{b^y + b^{-y}}$.

GROUP 60, p. 372

5. 3. **7.** 1, −2. **9.** $\dfrac{\log 5}{2 \log 3 - \log 5}$. **11.** $\ln (1 + \sqrt{2})$. **13.** ln 3.
15. $\dfrac{1}{2} \ln 2$. **17.** ln 2. **19.** −ln 2. **21.** $\dfrac{\log ra_n - \log a_1}{\log r}$.
23. $-CR \ln \left(\dfrac{CE - Q}{CE}\right)$. **25.** 4. **27.** 5. **29.** 3. **31.** 3. **33.** 2.
35. $x^2 - y^2 = 1$. **37.** $x^3 = ey^2$. **39.** $4x^2 + 4y^2 - z^2 = 0$.

GROUP 61, p. 380

1. 4.322. **3.** 1.167. **5.** 1.099. **7.** 2.332. **11.** 177.8. **13.** 1.695.
15. 2.894. **17.** 1.909. **19.** 159.7. **21.** 7791. **23.** 1.802. **25.** 0.2918.
27. 0.8096 sq ft. **29.** 12.95 sq in., 4.380 cu in. **31.** 16.98. **33.** 1.239 sec.
35. −1.529. **37.** 2.709. **39.** 1.946.

GROUP 62, p. 389

1. \$15. **3.** \$762.50. **5.** \$990.09. **7.** 3%. **9.** 6.19%. **11.** 25 yrs.
13. $2\frac{1}{2}$%. **17.** \$609. **19.** \$486.72. **21.** \$4438.55. **23.** \$2693.80.
25. 15.73 yrs. **27.** 14.07 yrs. **29.** $j = \left(1 + \dfrac{r}{n}\right)^n - 1;\ r = n[(1 + j)^{1/n} - 1]$.
33. 1.09344.

GROUP 63, p. 396

3. \$3210.81. **5.** \$6003.05; \$4055.45. **7.** \$1940.52; \$1625.16. **9.** \$1484.94.
11. \$50,000. **13.** \$197.20. **15.** 11. **17.** \$1844.11. **19.** \$122.89.
21. \$655.55. **29.** \$111.02.

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